Math 3215: Homework 4 answers

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1 Variance

1. You place a bet at the horse race. If your horse wins you make \$100, if he loses you lose \$5. Let's say the probability of him winning is .1.

   - What is your expected winngings? (in dollars). answ: $= .1 \cdot 100 - .9 \cdot 5 = 5.5$
   - What is the variance of your winnings? $= \mathbb{E}W^2 = (\mathbb{E}W)^2 = .1 \cdot 100^2 + .9 \cdot 25 - 5.5^2 = 1022.5 - 30.25 = 992.25$

2. You toss a fair coin 10 times. Let X be the number of times the coin came up differently than the last flip.

   - What is $\mathbb{E}X$? answ: $= 9 \cdot .5 = 4.5$.
   - What is $var(X)$? answ: $= 9 * .25 = 2.25$

3. In a room of 40 people, what is the expected number of pairs of people with the same birthday? answ: $= \binom{40}{2} \cdot \frac{1}{365} = 2.14$

4. What is the variance of the number of people with the same birthday? answ: $= \binom{40}{2}(1/365 - 1/365^2) = 2.13$

5. do the above question for n people in the room. $\mathbb{E}X = \binom{n}{2}/365$, $var(X) = \binom{n}{2}(1/365 - 1/365^2)$.

2 Statistics

1. Come up with an experiment, a null hypothesis, and an alternate hypothesis to answer the following questions with statistics:

   - Do more people prefer Pepsi or Coke? answ: Experiment: select people at random, have them blind taste test Coke and Pepsi (given to them in random order), ask which they prefer. Null 50% of people prefer Coke. Alternate: less than 50% of people prefer Coke.
   - Are there more blue-eyed people in Georgia or brown-eyed people? answ: Pick people from Georgia at random, record eye color. Null : same fraction blue as brown eyed. Alternate: fraction is not equal.
   - Are younger people more likely than older people to not have landlines?

2. Burger King claims that the average weight of their hamburgers is 3.2 oz after cooking.

   - If you measure 100 of them and the average is 2.8 oz, what can you conclude using statisitcs?
   - If you measure 10,000 of them and the average weight is 2.9 oz what can you conclude?

3 Continuous RV's

1. Let X and Y be independent random variables, each with a uniform distribution on $[0, 2]$.

   - What is the probability that both X and Y are larger ($>$) than 1.5?
     answ: $= \Pr[X > 1.5] \cdot \Pr[Y > 1.5] = 1/4 \cdot 1/4 = 1/16$
   - Both larger than or equal to 1.5? answ: same as above.
   - What's the probability $X + Y \geq 3$?
     answ:
     First find joint density function for X, Y. SInce they are independent, it is product of marginal density functions $f(x, y) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ on $[0, 2] \times [0, 2]$. Next we figure out what set we are integrating over, i.e. what event we want to know the probability of. It is the set $\{X + Y \geq 3\} = \{X, Y : X \in [1, 2], Y \in [1, 2] \text{ and} Y \geq 3 - X\}$. (if either is < 1 the suma can't be ≥ 3). So the probability is:

$$\int_1^2 \int_{3-x}^2 \frac{1}{4}\, dy\, dx$$
$$= \int_1^2 \frac{1}{4}(x - 1)\, dx$$
$$= \frac{1}{2} - \frac{1}{8} - \frac{1}{4} = \frac{1}{8}$$

   - What $\mathbb{E}(X + Y)$?
     answ: $= \mathbb{E}X + \mathbb{E}Y = 1 + 1 = 2$.
   - What's the variance of $(X + Y)$?
     answ: independent, so $var(X + Y) = var(X) + var(Y)$.

$$var(X) = \int_0^2 \frac{1}{2} x^2\, dx - 1$$
$$= 8/6 - 1 = \frac{1}{3}$$

     so $var(X + Y) = \frac{2}{3}$
   - What's $\Pr[X > 1.5 | X > 1]$?
     answ: using def. of conditional probability:

$$\Pr[X > 1.5 | X > 1] = \frac{\Pr[X > 1.5 \cap X > 1]}{\Pr[X > 1]}$$
$$= \frac{1/4}{1/2} = \frac{1}{2}$$

2. Let X have an exponential distribution: $f(x) = e^{-x}$, $x \geq 0$.

   - What's $\mathbb{E}X$?
     answ:

$$\mathbb{E}X = \int_0^\infty x e^{-x}\, dx = 1$$

     (integration by parts)

- What's $var(X)$?

$$\mathbb{E}X^2 = \int_0^\infty x^2 e^{-x}\, dx = 2$$

integration by parts twice. so $var(X) = 1$.

- What's the probability that $X > 5$?
answ: $= \int_5^\infty e^{-x} dx = e^{-5}$
- What's the probability that $X > 10$?
asnw: e^{-10}
- What's $\Pr[X > 5 | X > 10]$?
1 (I meant $\Pr[X > 10 | X > 5]$ which turns out to be e^{-5}.

3. Let X have an exponential distribution as above, and Y be uniform on $[-1, 1]$. Let X and Y be independent.

- What's $\Pr[X + Y \geq 0]$?

$$\begin{aligned}
&= \int_{-1}^{1} \int_{\max(0,-y)}^{\infty} \frac{1}{2} e^{-x}\, dx\, dy \\
&= 1 \cdot \Pr[Y \geq 0] + \frac{1}{2} \int_{-1}^{0} e^{y}\, dy \\
&= \frac{1}{2} + \frac{1}{2} - \frac{1}{2e} \\
&= 1 - \frac{1}{2e}
\end{aligned}$$

- What's $\mathbb{E}[X + Y]$?
answ: $= \mathbb{E}X + \mathbb{E}Y = 1 + 0 = 1$
- What's $\Pr[X + Y > 1 | X + Y > 0]$?
answ:

$$\begin{aligned}
&= \frac{\Pr[X + Y > 1]}{1 - \frac{1}{2e}} \\
&== \int_{-1}^{1} \int_{\max(1-y)}^{\infty} \frac{1}{2} e^{-x}\, dx\, dy / (1 - \frac{1}{2e}) \\
&= \frac{1}{2} \int_{-1}^{1} e^{y-1}\, dy / (1 - \frac{1}{2e}) \\
&= \frac{1 - \frac{1}{e^2}}{2 - \frac{1}{e}}
\end{aligned}$$