Math 3215: Lecture 11

Will Perkins

February 21, 2012

1 Law of Large Numbers

Just a reminder: we showed last week that if $X_1, \ldots X_n$ are i.i.d. random variables (what's iid?) and $S_n = \sum X_i$, then for any $\epsilon > 0$,

$$\Pr\left[\left| \frac{S_n}{n} - \mathbb{E}X_i \right| \right] \leq \frac{var(X_i)}{n\epsilon^2}$$

This is the Weak Law of Large Numbers. Another way to state this is:

$$\lim_{n\to\infty} \Pr\left[\left| \frac{S_n}{n} - \mathbb{E}X_i \right| \right] = 0$$

The Strong Law of Large Numbers is subtly different:

$$\Pr\left[\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}X_i \right] = 1$$

2 Continuous Probability Spaces

So far we have dealt with discrete probability: all our sample spaces have had a finite or countably infinite number of outcomes, each of which had probability > 0.

This is not always the right model however. Imagine picking a number between 0 and 1 at random, or modeling the height of a randomly selected person or the distance of a golf drive. In these cases we want all numbers (in some interval) to be possible, but if we set the probability of any particular number greater than 0 then the total probability would be infinite. So what can we do?

We can define probability entirely in terms of events instead of outcomes. The axioms of probability in terms of events are:

- $\Pr(A) \in [0, 1]$ for all events A
- $\Pr(S) = 1$
- $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ if $A \cap B = \emptyset$
- $\Pr(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$ if $A_i \cap A_j = \emptyset$ for all $i \neq j$

We'll look at a particular type of continuous probability space, where S can be represented as $\mathbb{R}$. In this case, events are subsets of real numbers, think intervals and unions of intervals.

What are possible probability functions?

Claim: the function

$$\Pr[A] = \int_A f(x)\, dx$$

satisfies the probability axioms if $f(x) \geq 0$ and $\int_{\mathbb{R}} f(x)\, dx = 1$.

Prove it!
Examples:

- $f(x) = 1$ for $x \in [0, 1]$, and 0 otherwise.
- $f(x) = e^{-x}$ for $x \geq 0$, and 0 otherwise.
- $f(x) = \frac{C}{1+x^2}$ for $x \geq 0$.
- $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. (a very important one)

Sketch $f(x)$ for the above.

3 Continuous Random Variables

Given a probability space on $\mathbb{R}$ as above, there is a natural random variable: $X(x) = x$. In that case we say X has *density function* $f(x)$. (This is the continuous analogue of the probability mass function for a discrete random variable).

Formulas:

- $F_X(t) = \Pr(X \leq t) = \int_{-\infty}^{t} f(x)\, dx$
- $\mathbb{E}X = \int_{\mathbb{R}} x f(x)\, dx$
- $\mathbb{E}(X^2) = \int_{\mathbb{R}} x^2 f(x)\, dx$

Examples: calculate the CDF, the expectation and the variance for the random variables with the following density functions:

1. $f(x) = e^{-x}$, $x \geq 0$
2. $f(x) = 1$ for $x \in [0, 1]$
3. $f(x) = 1/4$ for $x \in [-1, 3]$. (Is this a valid density function?)
4. $f(x) = 2$ for $x \in [1, 1.5]$. (Is this a valid density function?)
5. $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

The first example is an exponential random variable, the middle 3 are uniform random variables, the last is a Normal or Gaussian random variable.