Math 3215: Sample Midterm Questions

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Yes / no answers are not enough. Give some mathematical justification (which can be short and should not be an essay).

Answers like $\binom{10}{3}(.5)^6$ are fine - you don't need to give a decimal.

The questions on the actual midterm will vary in difficulty and in general the more difficult questions will count less.

1

I tell you that a coin has probability of heads .75. You flip it 4 times and get no heads. Do you believe my claim? Justify your answer with statistics.

answer:

We use the statistical method. Say we set $\alpha = .01$. Our null hypothesis, H_0 is that the coin has probability of heads .75. The alternate hypothesis could either be that $p \neq .75$ or that $p < .75$. In either case, we calculate the probability under the null hypothesis that what we saw occurs or something more extreme. In this case 4 tails in 4 flips is the most extreme outcome (4 heads is not as extreme since under H_0 the coin is biased in favor of heads.) $\Pr[4tails] = .25^4 = \frac{1}{256}$, so our p-value is less than .01 so we reject the null hypothesis and say that we don't believe the coin has $p = .75$.

2

We model tomorrow's weather as follows, with four possible outcomes: $(sunny, warm), (sunny, cold)$, $(rainy, warm), (rainy, cold)$ with probabilities $.2, .6, .1, .1$ respecitively. The salesman makes \$30 selling unmbrellas if it rains, but \$0 on umbrellas if it is sunny. He also makes \$60 selling gloves if it is cold, and \$10 selling gloves if it is warm.

Let Y be the total amount of money he makes, U be the amount he makes on umbrellas, and G the amount he makes on gloves.

- Calculate $\mathbb{E}Y, \mathbb{E}U, \mathbb{E}G$
- Calculate $var(Y)$
- Are U and G independent?

answer:

$$\mathbb{E}U = .2 \cdot 30 = 6$$

$$\mathbb{E}G = .7 \cdot 60 + .3 \cdot 10 = 45$$

$$\mathbb{E}Y = \mathbb{E}U + \mathbb{E}G = 51$$

3

Let X be a random variable that always takes a value ≥ 0. Prove that $\mathbb{E}X \geq \Pr[X \geq 1]$.

4

Prove that if X and Y are independent random variables then

$$\mathbb{E}[XY] = \mathbb{E}X \cdot \mathbb{E}Y$$

answer:

$$\begin{aligned}\mathbb{E}[XY] &= \sum_{t,s} ts \Pr[X = t, Y = s] \\ &= \sum_{t,s} ts \Pr[X = t] \cdot \Pr[Y = s] \\ &= \sum_{t} t \Pr[X = t] \mathbb{E}Y \\ &= \mathbb{E}Y\mathbb{E}X\end{aligned}$$

5

Let X be an non-negative, integer-valued random variable.

Prove that $\Pr[X \geq 1] \leq \mathbb{E}X$.

6

I pick a number from 1 to 1,000,000 at random, with equal probability.

- What's the probability the last digit is a 3 ?
- What's the probability the second-to-last digit is a 3?
- Are these events independent?
- What's the expected number of 3's in my number?

answers:

- 1/10
- 1/10
- probability both last and second to last are 3 is 1/100, so probabilities multiply and they are independent.
- $= 6 \cdot \frac{1}{10} = 3/5$

7

I have a class with 10 students.

1. How many different ways can I line them up, first to tenth?
2. If I pick a random way of lining them up, what's the probability the shortest student is in front?
3. What's the probability the tallest student is in back?
4. Are these events independent?

- 10!
- 1/10
- 1/10
- probability shortest in front and tallest in back $= \frac{1}{10} \cdot \frac{1}{9} \neq \frac{1}{10} \cdot 110$ so they are not independent.

8

In no more than 10 sentences, describe how you would use statistics to answer the following question:

"Does higher income lead to better health?"

After describing your procedure for answering that question, critique it: what might be the potential problems or biases in your method?

9

A policeman's radar gun says 'speeding' with probability .9 if a car is actually speeding. If the car is not speeding, it still will say 'speeding' with probability .01. 5% of drivers on the Garden Parkway are speeding. If The policeman points his radar gun at a random car and it says 'speeding' what's the chance the car is actually speeding?

10

Recall the geometric distribution: X is the number of tails before you get one head with a p-biased coin. i.e.

$$\Pr[X = i] = (1-p)^i p, \text{ for } i = 0, 1, \ldots$$

- Calculate $\mathbb{E}X$
- Calculate $var(X)$

11

You do the following. You flip a fair coin until you get a head. You then continue flipping until you get a tail after that. (So you flip the coin at least twice, and exactly twice only if you flip H, T. Another possible sequence is TTTHHHHT). Let X be the total number of flips, Y the total number of heads, Z the total number of tails.

- Caclulate $\mathbb{E}X, \mathbb{E}Y, \mathbb{E}Z$

12

Write the definition of two random variables X and Y being independent.

13

There is a lottery in which all 365 days of the year are put into a big hopper, and a random date drawn out. Whoever's birthday it is wins the lottery, and that date returned to the hopper.

The lottery happens 40 times, and only 1 of the winning days is a December birthday. The December birthdays are angry and think the lottery is biased.

What do you say? Justify your answer.

14

Let Y be the number of heads in n flips of a p-biased coin.

- What is $\mathbb{E}Y$?
- What is $var(Y)$?

15

Let X be a random variable, and let $Y = 2 - X$.

If $var(X) = 10$, what is $var(Y)$?

answer: $var(Y) = 10$, not independent because X is not constant and $\{X \leq t\}$ and$\{Y \geq 2 - t\}$ are the same event, so probabilities dont multiply.

16

Let A be the event that I get all heads in 6 flips of a p-biased coin, and B the event I get all tails.

1. Are A and B independent?
2. What is $\Pr[A|B^c]$?
3. What is $\Pr[A|B]$?
4. What is $\Pr[A \cap B]$?
5. What is $\Pr[A \cup B]$?

answers:

- no, disjoint $\Pr(A \cap B] = 0$.
- $= \frac{p^6}{1-(1-p)^6}$
- 0
- 0
- $p^6 + (1-p)^6$