Math 4221: Test #1 answers

September 14, 2012

1 Basic Probability

1.1

Let $\Omega = \mathbb{Z}$, the set of all integers, positive and negative. Give 3 different examples of sigma-fields on Ω.

A:

1. $\{\emptyset, \mathbb{Z}\}$
2. $\{\emptyset, \mathbb{Z}, evens, odds\}$
3. all subsets of $\mathbb{Z}$
4. $\{\emptyset, \mathbb{Z}, \{2,3\}, (\mathbb{Z} \setminus \{2,3\})\}$

1.2

Let $X_1, X_2, \ldots X_n$ be independent, identically distributed random variables with distribution function $F(x) = x$ for $x \in [0,1]$, $F(x) = 1$ for $x \geq 1$ and $F(x) = 0$ for $x \leq 0$. What is the probability that the maximum of $X_1, \ldots X_n$ is $\leq .9$?

A: $\{max \leq .9\} = \cap\{X_i \leq .9\}$. Since the X_i's are iid,

$$\Pr(\cap_i\{X_i \leq .9\}) = \Pr[X_i \leq .9]^n = .9^n$$

by the definition of the distribution function.

2 Expectation and Variance

2.1

You throw m balls independently and uniformly at random into n bins. Let X_i be the number of balls that land in bin i.

1. What is $\mathbb{E}(X_i)$?
2. What is $var(X_i)$?
3. What is $cov(X_i, X_j)$ for $i \neq j$?

A: $X_i \sim Bin(m, 1/n)$, so $\mathbb{E}(X_i) = \frac{m}{n}$ and $var(X_i) = \frac{m}{n} - \frac{m}{n^2}$.
To calculate the covariance we write

$$X_i = Y_{i1} + Y_{i2} + \cdots + Y_{im}$$
$$X_j = Y_{j1} + Y_{j2} + \cdots + Y_{jm}$$

where Y_{jk} is the indicator rv that ball k lands in bin j.

$$\mathbb{E}(X_i X_j) = \sum_{k_1,k_2} \mathbb{E}(Y_{ik_1} Y_{jk_2})$$

$\mathbb{E}(Y_{ik_1} Y_{jk_2}) = 0$ if $k_1 = k_2$ since the same ball cannot land in different bins at the same time, and $\mathbb{E}(Y_{ik_1} Y_{jk_2}) = \frac{1}{n^2}$ if $k_1 \neq k_2$ because different balls are independent. So

$$\mathbb{E}(X_i X_j) = m(m-1)/n^2$$

and

$$cov(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}X_i \mathbb{E}X_j = \frac{m^2}{n^2} - \frac{m}{n^2} - \frac{m^2}{n^2} = \frac{-m}{n^2}$$

A different way to calculate the covariance is to let $Z = \sum_{i=1}^{n} X_i$. Since Z is just the total number of balls we have $Z = m$ always, and as a random variable, $var(Z) = 0$. Now

$$var(Z) = \sum_i var(X_i) + \sum_{i \neq j} cov(X_i X_j)$$

By symmetry, each of the $n(n-1)$ covariance terms are equal, so we have the equation

$$0 = n \cdot var(X_i) + n(n-1) \cdot cov(X_i, X_j)$$

and solving for $cov(X_i, X_j)$ gives $\frac{-m}{n^2}$.

2.2

Say $\mathbb{E}(Z) = 5$ and $\mathbb{E}[(Z-5)^4] = 1$.

Give the best upper bound you can for $\Pr(\{X > 8\} \cup \{X < 2\})$.

A: Apply Markov's inequality with the non-negative function $f(x) = x^4$ to get:

$$\Pr[|Z - \mathbb{Z}| > t] \leq \frac{\mathbb{E}[(Z - \mathbb{E}Z)^4]}{t^4}$$

With $t = 3$, and $\mathbb{E}(Z) = 5$ we get

$$\Pr[|Z - 5| > 3] \leq \frac{1}{3^4} = \frac{1}{81}$$

3 Simple Random Walk

3.1

Find the asymptotics of $\Pr(S_n = 0)$ when $n \to \infty$ and n is even. (i.e. you should find a function $f(n)$ that is the product of powers of n, constants, and exponentials in n so that $\Pr(S_n = 0) \sim f(n)$. Your answer should not have factorials or binomial coefficients in it).

A: We'll use Stirling's formula in the third step:

$$\begin{aligned}
\Pr(S_n = 0) &= \binom{n}{n/2}(1/2)^n \\
&= \frac{n!}{(n/2)!(n/2)!}2^{-n} \\
&\sim \frac{n^n e^{-n}\sqrt{2\pi n}}{(n/2)^n e^{-n}\pi n}2^{-n} \\
&= \sqrt{\frac{2}{\pi n}}
\end{aligned}$$

3.2

Calculate the number of paths from $S_0 = 0$ to $S_{200} = 10$ that never return to 0.

A: A path that never returns to 0 must go up in the first step. Using the reflection principle, we know that the number of paths from $S_1 = 1$ to $S_{200} = 10$ that do hit 0 is equal to the total number of paths from $S_1 = -1$ to $S_{200} = 10$. We need 105 upsteps in 199 total steps to have such a path, and so the total number of paths that hit 0 is $\binom{199}{105}$. The total number of paths that go from $S_1 = 1$ to $S_{200} = 10$ is $\binom{199}{104}$, so the number that do not hit 0 is:

$$\binom{199}{104} - \binom{199}{105} = \frac{10}{200}\binom{200}{105}$$