Math 4221: Test #3

November 2, 2012

1 Branching Processes

1.1

Let Y be the offspring distribution of a Galton-Watson branching process and Z_n the number of individuals at generation n. Say $\mathbb{E}(Y) = 2/3$. Prove that $Z_n \to 0$ almost surely.

$\mathbb{E}Z_n = (2/3)^n$, so by Markov's Inequality $\Pr(Z_n \neq 0) \leq (2/3)^n$. Now since $\sum_{n=1}^{\infty}(2/3)^n < \infty$ (a geometric series), we can use Borel-Cantelli to say that with probability 1, $Z_n > 0$ only finitely many times. Thus $Z_n \to 0$ a.s.

1.2

Let Y be the offspring distribution of a branching process with $\Pr[Y = 0] = 1/3$ and $\Pr[Y = 2] = 2/3$. What is the probability that the branching process eventually becomes extinct?

Let p_e be the probability of extinction. We know

$$p_e = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot p_e^2$$

Solving this equation gives either $p_e = 1$ or $p_e = 1/2$, but we know that the probability of extinction is positive since the mean offspring distribution is > 1. So $p_e = 1/2$.

2 Generating and Characteristic Functions

2.1

Let $X \sim Bin(n,p)$ and $Y \sim Bin(X,q)$. What is the distribution of Y? Prove it.

$Y = y_1 + y_2 + \cdots + y_X$ where the y_i's are independent Bernoulli(q) rv's. (But we sum a random number of them). The generating function of Y is therefor

$$G_Y(s) = G_X(G_{y_i}(s))$$

then we calculate

$$G_X(G_{y_i}(s)) = [p \cdot (qs + 1 - q) + 1 - p]^n = [pqs + 1 - pq]^n$$

but this is the generating function of a Bin(n,pq) rv. So $Y \sim Bin(n,pq)$.

2.2

State and prove the weak law of large numbers assuming only a first moment. (i.e. cannot assume the random variables have a finite variance).

Statement: Let X_i's be iid rv's with $\mathbb{E}(X_i) = \mu$. Then

$$\frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{D} \mu$$

Proof: We calculate the characteristic function of $U_n = \frac{\sum_{i=1}^{n} X_i}{n}$:

$$\phi_{U_n}(t) = [\phi_{X_i}(t/n)]^n$$

Now we use a power expansion: $\phi_{X_i}(t/n) = \phi_{X_i}(0) + t/n\phi'_{X_i}(0) + o(t/n)$ to get:

$$\phi_{U_n}(t) = [1 + it/n\mu + o(t/n)]^n \to e^{it\mu} \text{ as } n \to \infty$$

And $\phi_\mu(t) = e^{it\mu}$ so the theorem is proved.