Math 4221: Test #4

November 30, 2012

1 Markov Chains

1.1

Write the transition matrix and find the stationary distribution of the following Markov Chain:

$$\begin{bmatrix} 0 & 1/3 & 2/3 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

The stationary distribution is $(1/3, 2/9, 4/9)$

1.2

Write down the definition of a reversible Markov chain. Is the above chain reversible?

A Markov chain with stationary distribution π is reversible if

$$\pi_i p_{ij} = \pi_j p_{ji}$$

for all i, j. (Think the mass sent from i to j equals the mass sent from j to i). The above chain is reversible.

2 Martingales

2.1

Let Z_n be a branching process with offspring mean > 1 and let p_e be its probability of extinction. Show that $M_n = (p_e)^{Z_n}$ is a martingale.

$$\begin{aligned}\mathbb{E}(M_n|\mathcal{F}_{n-1}) &= [\mathbb{E}p_e^Y]^{Z_{n-1}} \\ &= (G(p_e))^{Z_{n-1}} \\ &= p_e^{Z_{n-1}} = M_{n-1}\end{aligned}$$

A very nice solution one student had was to let X be the indicator rv that the branching process goes extinct. Then $M_n = \mathbb{E}[X|Z_{n-1}]$ is a Doob's martingale for X.

2.2

Let X be the number of triangles in the random graph $G(n,p)$ and let $\mathcal{F}$ be the sigma field generated by all edges containing vertex 1.

What is $\mathbb{E}[X|\mathcal{F}]$?

There are $\binom{n-1}{3}$ possible triangles not involving any edges containting vertex 1. Their expected number is $\binom{n-1}{3}p^3$ and is not affected by $\mathcal{F}$. Any triangle containing vertex 1 must contain two edges containing 1, and one edge joining the other two vertices. If Y is the number of edges containing 1, then the expected number of such triangles is $\binom{Y}{2}p$. So

$$\mathbb{E}[X|\mathcal{F}] = \binom{Y}{2}p + \binom{n-1}{3}p^3$$

A more formal way to solve this is to write X as the sum and product of the indicator random variables of all $\binom{n}{2}$ edges.

$$X = \sum_{i,j,k} Y_{ij}Y_{ik}Y_{jk}$$

where each Y_{ij} is 1 with probability p. Now we can take the expectation conditioned on $\mathcal{F}$ and use the rules of conditional expectation. Y_{1j} can be pulled out of the expectation since it is $\mathcal{F}$-measurable. For $i, j \neq 1$, $\mathbb{E}[Y_{ij}|\mathcal{F}] = \mathbb{E}[Y_{ij}]$ since the edges are all independent. This gives:

$$\mathbb{E}[X|\mathcal{F}] = \sum_{i,j\neq 1} Y_{1,i}Y_{1,j}p + \sum_{i,j,k\neq 1} p^3$$

$$= \binom{Y}{2}p + \binom{n-1}{3}p^3$$