Math 6221: Homework 2

Due February 5

1

Without writing anything down, calculate:

- The expected number of Aces in a hand of 5 cards
- The expected number of red card in a hand of 5 cards
- The expected number of people who get back their own coat, if all coats at a coat check are returned to random coat owners. [i.e. the expected number of fixed points of a random permutation on any number of elements]
- The expected number of back-to-back Heads in 100 flips of a fair coin.

2

Throw m balls uniformly and independently at random into n bins. Let X_i be the number of balls that land in bin i. Let Z be the number of empty bins.

- Calculate $\mathbb{E}X_i$, $var(X_i)$
- Calculate $cov(X_i, X_j)$ for $i \neq j$
- Calculate $\mathbb{E}Z$ and $var(Z)$
- Show that if $m = (1+\epsilon)n \log n$ then whp there are no empty bins
- Show that if $m = (1-\epsilon)n \log n$ then whp there is at least one empty bin

3

Let Z_1, Z_2 be independent standard normals. Let $X = aZ_1 + bZ_2$ and $Y = aZ_1 - bZ_2$.

- Calculate the expectation and variance of X and Y.
- Calculate $cov(X, Y)$
- Is there a choice of a, b that makes X and Y independent?

4

Prove that for a non-negative random variable Z,

$$\Pr[Z > 0] \geq \frac{(\mathbb{E}Z)^2}{\mathbb{E}(Z^2)}$$

Find an example where this inequality gives a better result than Chebyshev's Inequality.

5

Calculate the variance of Pois(λ) random variable

6

[Negative Correlation] Does the WLLN hold if instead of independence we have $cov(X_i, X_j) \leq 0$ for all $i \neq j$? If so, prove it, if not, give a counterexample.

7

Let $X_1, \dots X_n$ be independent uniform $[0, 1]$ rv's and let $Y = \max\{X_1, \dots X_n\}$. What is the distribution of Y?

8

Find a counting random variable Z_n with $\mathbb{E}Z_n \to \infty$ as $n \to \infty$, but $Z_n = 0$ whp.

9

[Birthday Problem] In a room of n people, what is the expected number of pairs of people that share the same birthday?

If we assume that the number of pairs follows a Poisson distribution (with the correct mean), what is the smallest n so that the probability that no people share a birthday is $< 1/2$?

How does this compare to the exact solution to the birthday problem?

What is the variance of the number of pairs that share a birthday? Is this what you'd expect given the above?