Math 6221: Homework 7

Due April 23

1

Flip a coin repeatedly. Let X_n = the number of heads up until flip n. Let Y_n be the longest consecutive run of heads up until time n. Let Z_n be the indicator that a head has come up by step n. Let Q_n be the indicator that back-to-back heads have come up by step n.

Which of the above are Markov Chains?

2

Find bounds on the mixing time of a random walk on a 2D and 3D grid of side length n. I.e. and $n \times n$ and $n \times n \times n$ grid.

3

Card Shuffling: Here is a method for shuffling a deck of n cards. At each step, pick a card uniformly at random and move it to the top of the deck.

1. Show that this defines an irreducible, aperiodic Markov chain on the space of all $n!$ orderings of the cards.
2. Is the chain reversible?
3. What is the stationary distribution?
4. Find bounds on the mixing time. Use a coupling argument for an upper bound.

4

Card shuffling part 2: Now order the cards $1 \dots N$. Now shuffle by picking two cards from the deck at random. If the lower card is higher in the deck, swap with probability p, and do nothing with probability $1 - p$. If the higher card is higher in the deck, swap with probability q, do nothing with probability $1 - q$. Assume $0 < p, q < 1$.

1. Show that this defines an irreducible, aperiodic Markov chain
2. Under what conditions is the uniform distribution on all $n!$ orderings stationary?

3. Find the stationaty distribution in general.
4. What bounds can you find on the mixing times?

5

Prove that a p-biased simple random walk on the integers is transient if and only if $p \neq 1/2$.

6

Prove that the largest eigenvalue of a stochastic matrix is 1.

7

Consider the Ehrenfest chain with 3 balls and two urns. Write the transition matrix. Find the eigenvalues and eigenvectors.

Can you generalize the eigenvalues to the case of n balls?

8

Describe a Markov Chain to sample a uniformly random matching in a given graph G (a matching is a collection of disjoint edges). Give the transition probabilities.

Now find a Markov Chain that samples a random matching but gives more weight to large matchings.