Branching Processes

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Galton and Watson

In 1873 Francis Galton wrote an article asking for an understanding of how family names in England would go extinct. A year later, Henry William Watson gave a mathematical solution and together they wrote a math paper introducing the mathematical frameowrk of the *branching process*.

Branching processes can be used to model:

- Philogenetic or family trees
- Atomic Chain Reactions
- BFS in a network
- Epidemics
- Rumor spreading

Galton-Watson Process

Definition

Let μ be a distribution on the non-negative integers. A Galton-Watson process with offspring distribution μ is a stochastic process with

$$Z_0 = 1$$

and

$$Z_n = X_{n,1} + X_{n,2} + \cdots + X_{n,Z_{n-1}}$$

where the sum is Z_{n-1} independent rv's each with distribution μ.

We think of Z_n as the number of individuals in generation n.

Markov Property

Exercise: Prove that Z_n is a homogeneous Markov Chain.

Which states are recurrent? Which are transient?

Extinction

A branching process 'goes extinct' if $Z_n = 0$ for some n.

We say a branching process 'survives' if it does not go extinct.

Warm-up

Say the offspring distribution has mean λ. What is $\mathbb{E}Z_n$?

Use conditioning.

$$\begin{aligned}\mathbb{E}Z_n &= \mathbb{E}[\mathbb{E}[Z_n|Z_{n-1}]] \\ &= \mathbb{E}[\lambda Z_{n-1}] \\ &= \lambda\mathbb{E}Z_{n-1}\end{aligned}$$

Then repeat the trick $n-1$ more times.

$$\mathbb{E}Z_n = \lambda^n$$

Extinction Probability

Show that if $\lambda < 1$ then the branching process goes extinct with probability 1.

Extinction Probability

What if $\lambda = 1$ or $\lambda > 1$?

Extinction Probability

Let y be the probability that a given branching process goes extinct. Let $p_k \Pr[X = k]$, the probability that one individual has k offspring. Then:

$$y = \sum_{k=0}^{\infty} p_k y^k$$

Why?

This looks familiar:

$$y = G_x(y)$$

The Generating Function of the offspring distribution!

Extinction Probability

So we want to solve $y = G_X(y)$.
There's always one solution to this equation: $y = 1$. But is that the only solution?

[Plot of a Poisson distribution with different means]

Extinction Probability

What conditions on the offspring distribution imply there are multiple solutions?

Which solution is the correct extinction probability?

Extinction Probability

Draw a picture and use Taylor's Theorem.

$$G_X(1) = 1$$

$$G'_X(1) = \mathbb{E}X$$

$$G_X(0) = p_0$$

This shows that if $\mathbb{E}X > 1$ then there are multiple solutions. If $\mathbb{E}X < 1$, there is only the single solution $y = 1$. If $\mathbb{E}X = 1$, there is a single solution as long as $p_0 > 0$ (and if not, we know the branching process is just $Z_n = 1$ for all n).

Extinction Probability

So it remains to determine which of the solutions is correct for $\mathbb{E}X > 1$.
We start by computing the generating function of Z_n:

$$G_{Z_1}(s) = G_X(s)$$

$$G_{Z_2}(s) = \mathbb{E}s^{Z_2} = \mathbb{E}[\mathbb{E}[s^{Z_2}|Z_1]] = \mathbb{E}[G_X(s)^{Z_1}] = G_X(G_X(s))$$

Extinction Probability

And iterating,

$$G_{Z_n}(s) = G_X(G_X(\cdots G_X(s)))$$

Let $y_n = \Pr[Z_n = 0]$. Then

1. y_n is increasing in n
2. $y_n \to y$ as $n \to \infty$
3. $y_n = G_X(G_X(\cdots G_X(0)))$

Extinction Probability

Theorem

The probability of extinction of a branching process is the smallest non-negative root of the equation:

$$y = G_X(y)$$

Proof: We have seen that the extinction probability satisfies the equation.

Let γ be some non-negative root of the equation. We claim that $y \leq \gamma$.

Extinction Probability

$$y_1 = G_X(0) \leq G_X(\gamma) = \gamma$$

since G is a non-decreasing function.

$$y_2 = G_X(y_1) \leq G_X(\gamma) = \gamma$$

and so on.. $y_n \leq \gamma$ for all n, and since $y_n \to y$, we have

$$y \leq \gamma$$

completing the proof.