Dependence and Conditioning

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Conditional Probability

Definition

If $\Pr(B) > 0$, then the conditonal probability of A given B is

$$\Pr[A|B] = \frac{\Pr(A \cap B)}{\Pr(B)}$$

What does this look like on a Venn diagram?

Conditional Distributions

We will discuss conditional distributions of random variables separately for discrete and continuous random variables. Later we will see a more general definition involving sigma-fields that encompasses both.

Discrete Random Variables

Let X be a discrete random variable and A some event.

Definition

The conditional probability mass function of X given A is:

$$f_{X|A}(x) = \Pr[X = x|A]$$

Definition

The conditional distribution function of X given A is:

$$F_{X|A}(t) = \Pr[X \leq t|A]$$

Discrete Random Variables

Using this, we can define:

Definition

The conditional expectation of X given A is

$$\mathbb{E}[X|A] = \sum_x x f_{X|A}(x)$$

The conditional expectation of a random variable given an event is a number, $\mathbb{E}(X|A)$.

Conditional Expectation

Often the event we condition on will be another random variable Y taking a specified value, i.e.

$$\mathbb{E}[X|Y=y] = \sum_x x \Pr[X=x|Y=y]$$

again, this is a number.

But we can also define the conditional expectation of X given Y as a random variable, and in particular, a function of Y.

Conditional Expectation

Let $f(y) = \mathbb{E}[X|Y = y]$. (This is a function $f : \mathbb{R} \to \mathbb{R}$). Then we define:

$$\mathbb{E}[X|Y] = f(Y)$$

So $\mathbb{E}[X|Y]$ is a random variable.

Conditional Expectation

Properties of conditional expectation:

1. $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$
2. Linearity: $\mathbb{E}[aX + bZ|Y] = a\mathbb{E}[X|Y] + b\mathbb{E}[Z|Y]$
3. $\mathbb{E}[\mathbb{E}[X|Y]g(Y)] = \mathbb{E}[Xg(Y)]$

Proof: ?

Continuous Random Variables

Conditioning on continuous random variables is a little more complicated since the event $Y = y$ has probability 0. We define:

Definition

For any y so that $f_Y(y) > 0$, we define the conditional density function of X given $Y = y$ as

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Similarly,

Definition

For any y so that $f_Y(y) > 0$, we define the conditional distribution function of X given $Y = y$ as

$$Y_{X|Y=y}(t) = \int_{\infty}^{t} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

Conditional Expectation

We can also define

Definition

The conditional expectation of a continuous rv X given a continuous rv $Y = y$ is

$$\mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y=y}(x)\, dx$$

And considering the above as a function $g(y)$, we define the random variable

$$\mathbb{E}[X|Y] = g(Y)$$

just as in the discrete case. The same properties hold.

Conditioning on Multiple Random Variables

We can also define

$$\mathbb{E}[X|Y_1, Y_2, \dots Y_k]$$

For discrete RV's, this is

$$\mathbb{E}[X|Y_1, Y_2] = \sum_x f_{X|,Y_1,Y_2}(x)$$

Where $f_{X|Y_1,Y_2}(x)$ is a function that depends on x and also on the values of Y_1, Y_2.

The conditional expectation is your 'best guess' of X given the infomation of the values of Y_1, Y_2.

Again, it is a random variable, but becomes a number when we specify the particular values of Y_1 and Y_2.

Examples

Choose a point uniformly at random in the unit square. Let X be its x-coordinate, Y its y-coordinate, and $R = X^2 + Y^2$.

1. Find the joint density function of X and R
2. Find the conditional density function of X given $R = 1$
3. Find the conditional expectation of X given R

Examples

Let $p \sim Unif[0,1]$ and $X \sim Bin(n,p)$.

1. Find $\mathbb{E}[p|X]$.
2. Find $\mathbb{E}[X,p]$

Examples

Let S_n be a simple symmetric random walk. Define the conditional process $M_n(k)$ as the random walk conditioned on $S_{100} = k$. What is the distribution of this process?

An Application

We saw that

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$$

This can be a useful formula for calculating expectations.

Simple example: Let $p \sim Unif[0,1]$, $X \sim Bin(n,p)$.
What is $\mathbb{E}X$?

A Recursive Example

Let $Z_0 \sim Pois(\lambda)$.
Let $Z_1 \sim Pois(Z_0)$.
....
Let $Z_n \sim Pois(Z_{n-1})$.

Calculate $\mathbb{E}Z_n$.

Another Example

Let S_n be a simple random walk.
Fix N and for $k \leq N$, let $M_k = \mathbb{E}[S_N | S_0, S_1, \dots S_k]$.

What is M_k?
Does your answer change if S_n is not a symmetric random walk?