# First-Moment Method

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# Markov's Inequality

**Theorem**

*Markov's Inequality Let $X$ be a non-negative random variable. Then*

$$\Pr[X \geq t] \leq \frac{\mathbb{E}X}{t}$$

# Counting Random Variables

Let $X$ be a non-negative integer rv, i.e. a counting random variable. Then setting $t = 1$,

$$\Pr[X \neq 0] \leq \mathbb{E}X$$

In particular, if $\mathbb{E}X = o(1)$, then we can conclude that $X = 0$ with probability $1 - o(1)$.
Typical application: To show that no 'bad events' happen', show that the expected number of bad events is small.

# The Method

Outline of the Method:

1. Want to show that whp no 'bad events' happen.
2. Let $X$ be the number of bad events that occur.
3. Write $X = X_1 + X_2 + \cdots + X_n$ as the sum of indicator rv's, where $X_i = 1$ if bad event $i$ occurs.
4. $\mathbb{E}X_i = p_i = \Pr[B_i \text{ occurs}]$
5. By linearity, $\mathbb{E}X = \sum \mathbb{E}X_i = \sum p_i$
6. By Markov's Inequality, $\Pr[X > 0] \leq \mathbb{E}X$.
7. If $\mathbb{E}X = \sum p_i = o(1)$, then conclude that $X = 0$ (i.e. no bad events occur) with high probability.

# Example 1

Show that with high probability, a simple, symmetric random walk does not cross 0 between steps $n$ and $n + n^{1/3}$.

Proof: Let $X$ be the number of times $S_k = 0$ for $k \in [n, n + n^{1/3}]$.

Calculate $\mathbb{E}X$, show that it $\to 0$ as $n \to \infty$.

$$\begin{aligned}
\mathbb{E}X &= \sum_{k=n}^{n+n^{1/3}} \Pr[S_k = 0] \\
&= O\left(n^{1/3} n^{-1/2}\right) \\
&= o(1)
\end{aligned}$$

Show that for any $\epsilon > 0$, the maximum of $n$ standard normal random variables is $\leq (1+\epsilon)\sqrt{2 \log n}$ whp.

Q: Do we need the rv's to be independent?
A: No! Dependency is irrelevant for first-moment method. This makes the method very useful. Notice that we very often use the linearity property of expectation in computing the first-moment.

Use Normal tail bound to prove.

Throw $m$ balls uniformly and independently at random into $n$ bins. Show that if $m > (1+\epsilon) n \log n$, whp there are no empty bins.

Let $X$ be the number of empty bins. Then

$$\begin{aligned}\mathbb{E}X &= \sum_{i=1}^{n} \Pr[\text{ bin i empty}] \\ &= n \cdot \left(1 - \frac{1}{n}\right)^m \\ &\sim n \cdot e^{-(1+\epsilon)\log n} \\ &\sim n^{-\epsilon} \to 0\end{aligned}$$

So with probability $1 - o(1)$, there are no empty bins.

Q: What if $m = (1 - \epsilon) n \log n$?
A: Stay tuned for the Second-Moment Method!