Generating Functions

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Turning a Function into a Sequence

Definition

Let $\mathbf{a} = a_0, a_1, a_2, \ldots$ be a sequence of real numbers. Then the generating function of $\mathbf{a}$ is

$$G_a(x) = a_0 + a_1x + a_2x^2 + \ldots$$

This is called a 'formal power series' since we define the function without worrying whether or not the series converges. (It may for some choices of x but not for others).

Basic Examples

The sequence $\mathbf{a} = 1, 1, \ldots$ has the generating function

$$G(s) = 1 + s + s^2 + \cdots = \frac{1}{1-s}$$

The sequence $\mathbf{a} = 1, q, q^2 \ldots$ has the generating function

$$G(s) = \frac{1}{1-qs}$$

The sequence $\mathbf{a} = 1/0!, 1/1!, 1/2!, \ldots$ has the generating function

$$G(s) = e^s$$

Probability Generating Functions

Let X be a discrete random variable taking the values $0, 1, 2, \ldots$. Then there is a generating function easily associated to X:

$$G_X(s) = \Pr[X = 0] + \Pr[X = 1]s + \Pr[X = 2]s^2 + \ldots$$

This is the probability generating function of X.

Examples - find the generating functions for:

- Bernoulli
- Poisson
- Discrete Uniform
- Geometric
- Binomial (later)

Properties of Probability Generating Functions

Some properties of $G_X(s)$:

1. $G_X(1) = 1$
2. $G_X(0) = \Pr[X = 0]$
3. $G'_X(0) = \Pr[X = 1]$
4. $G'_X(1) = \mathbb{E}X$
5. $G_X^{(k)}(0) = ?$
6. $G_X^{(k)}(1) = ?$
7. $G_X(s) = \mathbb{E}s^X$

Convolutions

Definition

Let $\mathbf{a} = a_0, a_1, \ldots$ and $\mathbf{b} = b_0, b_1, \ldots$. The convolution of $\mathbf{a}$ and $\mathbf{b}$, denoted $\mathbf{a} * \mathbf{b}$ is the sequence $c_0, c_1, c_2, \ldots$ in which

$$c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

I.e.,

$$a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, \ldots$$

Convolutions

Fact:
If π_X and π_Y are the probability mass sequences of two independent discrete random variables X and Y, then the probability mass sequence of $X + Y$ is

$$\pi_{X+Y} = \pi_X * \pi_Y$$

Proof:

$$\Pr[X + Y = k] = \sum_{j=0}^{k} \Pr[X = j] \Pr[Y = k - j]$$

Convolutions and Generating Functions

Convolutions work nicely with generating functions:

$$G_{a*b} = G_a \cdot G_b$$

Examples

Prove that the sum of two independent Poisson RV's is another Poisson.

Find the generating function of a binomial RV.

Composition of Generating Functions

We can use generating functions to understand sums of a random number of independent random variables:

Theorem

Let $Z = X_1 + \cdots + X_N$ where the X_i's are iid with generating function G_X and N is a random variable with generating function G_N. Then

$$G_Z(s) = G_N(G_X(s))$$

Proof: Using conditional expectation.

Example

Let $N \sim Pois(\lambda)$ and let $X \sim Bin(N, p)$. What is the distribution of X?