Independence

Will Perkins

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Definition

Two events A and B are *independent* if:

$$P(A \cap B) = P(A)P(B)$$

Prototypical example: coin flips. Check that two different coin flips are independent.

Warning: Independence is very different that disjointness!
$(A \cap B = \emptyset)$.

Definition

A collection of events $A_1, A_2, \ldots A_n$ are independent if

$$P\left(\cap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i)$$

for any subset $I \subseteq \{1, \ldots n\}$

Events can be pairwise independent but not independent!
Example: Flip two fair coins. A is the event that the first flip is a head, B the event that the second flip is a head, and C the event that both flips are the same. Show that these events are pairwise independent but not jointly independent.

Definition

Two random variables X and Y are independent if $\{X \in A\}$ and $\{Y \in B\}$ are independent events for all Borel sets A and B.

Fact: enough to check for sets of the form $(-\infty, t]$.

Definition

Two σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ are independent if

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

for any $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2$.

Independence greatly simplifies the joint distribution of random variables. We say the independent random variables have the *product measure* as their joint distribution. Why?

Let $R = E_1 \times E_2$ be a rectangle in $\mathbb{R}^2$. Then if

$$\mu(R) = \mu_1(E_1) \cdot \mu_2(E_2)$$

for all (generalized) rectangles R, then we say μ is the product of the measures μ_1 and μ_2.

This is exactly the same as saying that μ is the joint distribution of *independent* random variables X_1 and X_2 with distributions μ_1 and μ_2 respectively.

Independent Random Variables

What does this mean for calculating things? If random variables (or events) are independent, you multiply to get the probability of the intersection ('AND').

If X, Y are independent,

$$\Pr[X \le t \cap Y \le s] = \Pr[X \le t] \cdot \Pr[Y \le s]$$

If X, Y are discrete and independent,

$$\Pr[X = t \cap Y = s] = \Pr[X = t] \cdot \Pr[Y = s]$$

If X, Y are continuous and independent,

$$f_{X,Y}(t, s) = f_X(s) f_Y(t)$$

Let X and Y be independent standard normal random variables.

- What is their joint density function?
- On what probability space are they defined?

Often we will want to define an infinite sequence of independent random variables $X_1, X_2, \ldots$.
Is this even possible?

What is the sample space?
What is the sigma field?
What is the probability measure?

First example: infinite sequence of independent coin flips.

Kolmogorov Extension Theorem

How about a general solution?

- Want to be able to ask things like "What's the probability the first flip is a head?" Or, "What's the probability we get at leat 50 heads in the first 100 flips?"
- So we define $\mathcal{F}$ to be the sigma field generated by all of these finite-dimensional cyllindar sets. (Events that only depend on the first K flips, for every constant K).
- We know the probabiluty measure we want on these sets - product measure.
- Kolmogorov says that this measure can be uniquely extended to a measure on the whole σ-field.

Sums of Independent Random Variables

The two main theorems in this course will be concerned with the sums of independent random variables. What is the distribution of the sum of two (or more) independent random variables?

Let X, Y be independent, and let $Z = X + Y$. In terms of the distributions of X and Y, what is the distribution of Z?

$$\{Z \leq t\} = \bigcup_{s} \{X \leq s \cap Y \leq t - s\}$$

$$F_Z(t) = \Pr[Z \leq t] = \int F_X(t - s) \, d\mu_Y(s) = \int F_Y(t - s) \, d\mu_X(s)$$

We write $\mu_{X+Y} = \mu_X * \mu_Y$ where $*$ is *convolution*.

The previous formula simplifies in the case of discrete or continuous random variables:

Discrete (convolution of probability mass functions):

$$f_{X+Y}(t) = \sum_s f_X(t - s) f_Y(s)$$

Continuous (convolution of probability density functions):

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(t - s) f_Y(s) \, ds$$

Let $X \sim Pois(\mu)$ and $Y \sim Pois(\lambda)$ be independent.
Find the distribution of $X + Y$.

Let $X, Y \sim$ *Uniform*$[0, 1]$ be independent.
Find the distribution of $X + Y$.

Let $X, Y \sim Exponential(\mu)$ be independent.
Find the distribution of $X + Y$. [careful: note that f_X and f_Y are only defined on $[0, \infty)$.]

Let $X, Y \sim N(0, 1)$ be independent.
Find the distribution of $X + Y$.

Let $X \sim Bin(n, p)$ and $Y \sim Bin(m, p)$ be independent.
Find the distribution of $X + Y$.