Abstract Measures and Integration

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January 9, 2013

Sigma Fields

Recall that a σ-field $\mathcal{F}$ is a collection of subsets of a sample space Ω so that:

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
3. if $A_1, A_2, \cdots \in \mathcal{F}$, then $\cup A_i \in \mathcal{F}$

Measures

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- $\mu(\emptyset) = 0$
- If $A_1, A_2, \ldots$ are disjoint, then

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Probability Measures

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Exercise: How can you characterize all probability measures on $(\mathbb{R}, \mathcal{B})$? Give several examples of different probability measures on $\mathbb{R}$.

Riemann Integration

What is the Riemann Integral and why do we need a new one?

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Small problem: some (strange) functions are not integrable.
eg. Let $f(x) = 1$ if x is rational, 0 if x is irrational. Prove that f is not Riemann integrable.

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Big problem: Riemann integral has a problem with limits.
Pointwise limit of a sequence f_n of Riemann integrable functions may not be integrable!
(that's a bad thing..)

Measurable Functions

Definition

A *measurable function* $f : \Omega \to \mathbb{R}$ on $(\Omega, \mathcal{F})$ is a function so that $f^{-1}(U) \in \mathcal{F}$ for every open set $U \subseteq \mathbb{R}$.

Analysis Background

Need to know some definitions from analysis:

- sup, inf, $\limsup$, $\liminf$

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In the HW: Prove that if f_n is a sequence of measurable functions then $\limsup f_n, \liminf f_n, \sup f_n, \inf f_n$ are all measurable functions.

Indicator Functions

Definition

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When is an indicator function a measurable function?

Simple Functions

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$f(x)$ is measurable if $A_i \in \mathcal{F}$ for all i.

The Integral

Definition

Let $f = \sum_{i=1}^{N} \alpha_i \mathbf{1}_{A_i}$ be a non-negative, measurable simple function. Then the integral of f with respect to the measure P is defined as:

$$\int f = \sum_{i=1}^{N} \alpha_i P(A_i)$$

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Definition

A non-negative measurable function f is *integrable* if $\int f < \infty$.

The Integral

Let f be a measurable function.
Define $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$.
Then $f(x) = f^+(x) + f^-(x)$ and we define

$$\int f = \int f^+ - \int f^-$$

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Definition

A measurable function f is integrable if both f^+ and f^- are integrable.

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5. $\int (f+g) = \int f + \int g$
6. Jensen's Inequality: If $g : \mathbb{R} \to \mathbb{R}$ is convex, then

$$g\left(\int f\right) \leq \int g(f)$$

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Set $A_0 = (\cup A_i)^c$ and $B_0 = (\cup B_i)^c$

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How to extend for general measurable functions?

Monotone Convergence Theorem

Theorem

Suppose $f_n \geq 0$, $f_n(x) \to f(x)$ almost surely and $f_n(x)$ is non-decreasing in n. Then

$$\lim_{n \to \infty} \int f_n = \int f$$

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Proof?

Fatou's Lemma

Theorem

If $f_n \geq 0$*, then*

$$\liminf_{n\to\infty} \int f_n \geq \int (\liminf_{n\to\infty} f_n)$$

Dominated Convergence Theorem

Theorem

Suppose $|f_n(x)| \leq g(x)$ *for all* x *and* n, $f_n(x) \to f(x)$ *a.s., and* $\int g < \infty$. *Then*

$$\lim_{n\to\infty} \int f_n = \int f$$

Counterexamples

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Example 2: Let $f_n = \frac{1}{n}\mathbf{1}_{[-n,n]}$. Then $\int f_n = 2$ for all n, but $f_n(x) \to 0$ a.s.