Janson's Inequality, Local Lemma

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Janson's Inequality

First a detour back to Poisson convergence.
HW problem (modified): If $p = n^{-2/3}(\log n)^{1/3}$, show that the number of vertices that are not in any triangle has a Poisson distirbution.

It's tricky enough just to compute the expectation.

Janson's Inequality

Setting: Large 'ground set' $\mathcal{R}$, take a random set $S \subset \mathcal{R}$ where each $r \in \mathcal{R}$ is in S with probability p_r independently.

Let $\{A_1, \ldots A_m\}$ be a collection of subsets of $\mathcal{R}$. And let B_i be the 'bad event' that $A_i \subseteq S$ - i.e. all elements of A_i appear in the random set.

We want bounds on the probability that no bad event happens.

Janson's Inequality

What if all the A_i's were disjoint? Then the bad events would be independent, and if X is the number of bad events,

$$\Pr[X = 0] = \prod_i \Pr[B_i^c]$$

We want to understand how dependent the events can be and still get a bound close to this.

Janson's Inequality

Let $\mu = \mathbb{E}X$.

$$\mu = \sum_i \Pr[B_i]$$

Notice

$$\prod_i \Pr[B_i^c] \leq e^{-\mu}$$

(from the inequality $1 - x \leq e^{-x}$) and often we will have

$$\prod_i \Pr[B_i^c] \sim e^{-\mu}$$

Janson's Inequality

What about dependencies? Let $i \sim j$ if A_i and A_j intersect (i.e. B_i and B_j are dependent).
Define

$$\Delta = \sum_{i \sim j} \Pr[B_i \wedge B_j]$$

Here the sum is over ordered pairs i, j.

Janson's Inequality

Theorem (Janson's Inequality)

With the set-up as above,

$$\prod_i \Pr[B_i^c] \leq \Pr[X = 0] \leq e^{-\mu + \Delta/2}$$

Example

A quick example: What is the probability that there are no triangles in $G(n, p)$ when $p = n^{-4/5}$?

With Chebyshev we would get something. We can't apply Chernoff bounds because the triangles are not independent (we could look at a set of disjoint triangles but there are not enough of them).

1) Check that the above set-up applies.
2)

$$\mu = \binom{n}{3} p^3 \sim n^{3/5}/6$$

Example

$$\Delta = \sum_{i \sim j} \Pr[B_i \wedge B_j] = ?$$

Fix a triangle. There are $3(n-3)$ triangles that share an edge with it. The probability that both triangles are present is p^5. So

$$\Delta = \binom{n}{3} 3(n-3)p^5 \sim n^4 p^5 / 2$$

Janson's inequality gives:

$$(1-p^3)^{\binom{n}{3}} \leq \Pr[X=0] \leq e^{-\binom{n}{3}p^3 + n^4 p^5/2}$$

Example

Note that for $p = n^{-4/5}$, $n^4 p^5 = o(n^3 p^3)$, so

$$\Pr[X = 0] \sim e^{-n^3 p^3/6} = e^{-n^{3/5}/6}$$

This 'works' up until $n^3 p^3 = n^4 p^5$, i.e. $p = n^{-1/2}$.

Lovasz Local Lemma

Here's another nice probabilistic tool.

A simple observation: If a finite collection of events is independent and each has probability less than 1, then there is a positive probability that none of the events happen.

But what if the events have some dependence?

Lovasz Local Lemma

Theorem

Let $A_1, \ldots A_n$ be events with a dependency graph that has maximum degree d. Suppose $\Pr[A_i] \leq p$ for all i. Then if

$$ep(d+1) \leq 1$$

there is a positive probability that no events occur.

Example

Theorem

Any k-CNF formula in which no variable appears in more than $2^{k-2}/k$ clauses is satisfiable.

Proof

Claim: for any $S \subset \{1, \ldots n\}$,

$$\Pr\left[A_i | \bigcap_{j \in S} A_j^c\right] \leq \frac{1}{d+1}$$

The theorem follows from the claim by using the chain rule:

$$\Pr\left[\bigcap_i A_i^c\right] = \prod_{i=1}^{n}\left(1 - \Pr[A_i | \bigcap_{j<i} A_j^c]\right) \geq \left(1 - \frac{1}{d+1}\right)^n > 0$$

Proof

Proof of the claim: induction of the size of S. For $S = \emptyset$, use the condition $p \leq \frac{1}{(d+1)e}$. Now separate S into S_1, coordinates j so that $i \sim j$, and S_2, coordinates j so that A_i and A_j are independent.

Then write

$$\Pr\left[A_i | \bigcap_{j \in S} A_j^c\right] = \frac{\Pr[A_i \cap \bigcap_{j \in S_1} A_j^c | \bigcap_{j \in S_2} A_j^c}{\Pr[\bigcap_{j \in S_1} A_j^c | \bigcap_{j \in S_2} A_j^c}$$

$$\leq \frac{\Pr[A_i}{(1 - 1/(d+1))^d}$$

why ?

$$\leq ep \leq \frac{1}{d+1}$$