Janson’s Inequality, Local Lemma

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First a detour back to Poisson convergence.
HW problem (modified): If $p = n^{-2/3}(\log n)^{1/3}$, show that the number of vertices that are not in any triangle has a Poisson distribution.

It’s tricky enough just to compute the expectation.
Setting: Large ‘ground set’ $\mathcal{R}$, take a random set $S \subset \mathcal{R}$ where each $r \in \mathcal{R}$ is in $S$ with probability $p_r$ independently.

Let \{\(A_1, \ldots, A_m\)\} be a collection of subsets of $\mathcal{R}$. And let $B_i$ be the ‘bad event’ that $A_i \subseteq S$ - i.e. all elements of $A_i$ appear in the random set.

We want bounds on the probability that no bad event happens.
What if all the $A_i$’s were disjoint? Then the bad events would be independent, and if $X$ is the number of bad events,

$$\Pr[X = 0] = \prod_{i} \Pr[B_i^c]$$

We want to understand how dependent the events can be and still get a bound close to this.
Janson’s Inequality

Let $\mu = \mathbb{E}X$.

$$\mu = \sum_i \Pr[B_i]$$

Notice

$$\prod_i \Pr[B_i^c] \leq e^{-\mu}$$

(from the inequality $1 - x \leq e^{-x}$) and often we will have

$$\prod_i \Pr[B_i^c] \sim e^{-\mu}$$
What about dependencies? Let \( i \sim j \) if \( A_i \) and \( A_j \) intersect (i.e. \( B_i \) and \( B_j \) are dependent).
Define

\[
\Delta = \sum_{i \sim j} \Pr[B_i \land B_j]
\]

Here the sum is over ordered pairs \( i, j \).
Theorem (Janson’s Inequality)

With the set-up as above,

$$\prod_{i} \Pr[B_{i}^{c}] \leq \Pr[X = 0] \leq e^{-\mu + \Delta/2}$$
A quick example: What is the probability that there are no triangles in $G(n, p)$ when $p = n^{-4/5}$?

With Chebyshev we would get something. We can’t apply Chernoff bounds because the triangles are not independent (we could look at a set of disjoint triangles but there are not enough of them).

1) Check that the above set-up applies.
2) 

$$\mu = \binom{n}{3} p^3 \sim n^{3/5}/6$$
Example

\[ \Delta = \sum_{i \sim j} \Pr[B_i \land B_j] =? \]

Fix a triangle. There are \(3(n - 3)\) triangles that share an edge with it. The probability that both triangles are present is \(p^5\). So

\[ \Delta = \binom{n}{3} 3(n - 3)p^5 \sim n^4 p^5 / 2 \]

Janson’s inequality gives:

\[ (1 - p^3) \binom{n}{3} \leq \Pr[X = 0] \leq e^{-\binom{n}{3} p^3 + n^4 p^5 / 2} \]
Note that for $p = n^{-4/5}$, $n^4 p^5 = o(n^3 p^3)$, so

$$\Pr[X = 0] \sim e^{-n^3 p^3 / 6} = e^{-n^{3/5} / 6}$$

This ‘works’ up until $n^3 p^3 = n^4 p^5$, i.e. $p = n^{-1/2}$. 

Example
Here's another nice probabilistic tool.

A simple observation: If a finite collection of events is independent and each has probability less than 1, then there is a positive probability that none of the events happen.

But what if the events have some dependence?
Let $A_1, \ldots, A_n$ be events with a dependency graph that has maximum degree $d$. Suppose $\Pr[A_i] \leq p$ for all $i$. Then if

$$ep(d + 1) \leq 1$$

there is a positive probability that no events occur.
Theorem

Any $k$-CNF formula in which no variable appears in more than $2^{k-2}/k$ clauses is satisfiable.
Proof

Claim: for any $S \subset \{1, \ldots n\}$,

$$\Pr \left[ A_i \mid \bigcap_{j \in S} A_j^c \right] \leq \frac{1}{d + 1}$$

The theorem follows from the claim by using the chain rule:

$$\Pr \left[ \bigcap_{i} A_i^c \right] = \prod_{i=1}^{n} \left( 1 - \Pr[A_i \mid \bigcap_{j < i} A_j^c] \right) \geq \left( 1 - \frac{1}{d + 1} \right)^n > 0$$
Proof

Proof of the claim: induction of the size of $S$. For $S = \emptyset$, use the condition $p \leq \frac{1}{(d+1)e}$. Now separate $S$ into $S_1$, coordinates $j$ so that $i \sim j$, and $S_2$, coordinates $j$ so that $A_i$ and $A_j$ are independent.

Then write

$$\Pr \left[ \frac{A_i}{\cap_{j \in S} A_j^c} \right] = \frac{\Pr[ A_i \cap \cap_{j \in S_1} A_j^c | \cap_{j \in S_2} A_j^c ]}{\Pr[ \cap_{j \in S_1} A_j^c | \cap_{j \in S_2} A_j^c ]}$$

$$\leq \frac{\Pr[A_i]}{(1 - 1/(d+1))^d}$$

why?

$$\leq ep \leq \frac{1}{d+1}$$