Poisson Convergence

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Back to the Birthday Problem

On HW # 2, you computed the expectation and variance of the number of pairs of people with the same birthday in a room of n people.

$$\mathbb{E}Z = \binom{n}{2}\frac{1}{365}$$

$$\text{var}(Z) = \binom{n}{2}\left(\frac{1}{365} - \frac{1}{365^2}\right)$$

If you compute these, you'll see that they are close together. Z is also a counting random variable, i.e. a non-negative integer. Another way to look at it is that Z is the number of (nearly independent) 'rare' events that occur.

Back to the Birthday Problem

In these cases we would like to say that Z is nearly a Poisson random variable with mean $\mathbb{E}Z$.

In this case, we get a very good approximation by assuming Z is Poisson.

$$\Pr[\mathit{Pois}(\binom{23}{2}\frac{1}{365}) \geq 1] = .500002$$

vs

$$Pr[Z \geq 1] = .507297$$

The Law of Small Numbers

A good general rule is:
If X is the number of a large collection of potential rare and nearly independent events that occur, then

$$X \approx Pois(\mathbb{E}X)$$

and in particular, $\Pr[X = 0] \sim e^{-\mathbb{E}X}$.

But when does this hold?

The Method of Moments

Strange Fact: Two random variables X and Y can have the same moments: $\mathbb{E}X^k = \mathbb{E}Y^k$ for all k, yet have different distirbutions. However, certain distributions are *determined by their moments*. I.e. they are the only distributions with that sequence of moments. Examples including Normal and Poisson distributions.

If X is a distribution determined by its moments, then if

$$\mathbb{E}X_n^k \to \mathbb{E}X^k$$

for all k, then $X_n \xrightarrow{d} X$.

Poisson Convergence

Let $B_1, B_2, \ldots B_n$ be a sequence of 'Bad' events. Let X_i be the indicator RV of B_i and let $X = \sum X_i$ be the number of bad events that occur.

- Suppose that $\mathbb{E}X \to \mu$ as $n \to \infty$
- Suppose that for all constant r,

$$\sum_{i_1,\ldots i_r} \Pr[B_{i_1} \cap \cdots \cap B_{i_r}] \to \frac{\mu^r}{r!}$$

Then $X \to Pois(\mu)$, and in paritcular, $\Pr[X = k] \to e^{-\mu}\frac{\mu^k}{k!}$.

Proof

Bonferroni Inequalities: the Inclusion/Exclusion probabilities are alternatingly over and under-estimates.

$$\Pr[X=0] \le 1 - \sum_{i} \Pr[B_i] + \sum_{i,j} \Pr[B_i \wedge B_j] - \cdots + \sum_{i_1,\dots i_r} \Pr[B_{i_1} \wedge \cdots \wedge B_{i_r}]$$

where r is even.

Similarly,

$$\Pr[X=0] \ge 1 - \sum_{i} \Pr[B_i] + \sum_{i,j} \Pr[B_i \wedge B_j] - \cdots - \sum_{i_1,\dots i_r} \Pr[B_{i_1} \wedge \cdots \wedge B_{i_r}]$$

where r is odd.

Proof

Fix an ϵ. Using Taylor series you can show that for large enough R,

$$\left| \sum_{r=0}^{R} (-1)^r \frac{\mu^r}{r!} - e^{-\mu} \right| < \epsilon$$

Now apply Bonferroni's Inequalities, and let $n \to \infty$ so that

$$\left| \sum_{i_1,\ldots i_r} \Pr[B_{i_1} \cap \cdots \cap B_{i_r}] - \frac{\mu^r}{r!} \right| < \epsilon$$

for $r < R$.

Examples

- n people give their hats to a hat check but the hats are returned at random. Show that the number of people who get their own hat back is approximately Poisson.
- Apply the method to the Birthday Problem.
- Consider a random graph $G(n, p)$ with $p = \frac{\log n + c}{n}$. Show that the number of isolated vertices follows a Poisson distribution.

Dependency Graphs

Sometimes the situation is much more complicated and it's difficult to compute the above probabilities.

A *Dependency Graph* is a set of nodes and edges where:

- The nodes represent events B_i
- A node B_i is connected to another node B_j if B_i and B_j are dependent.
- The neighborhood of a node is the set of all events B_i depends on.

Dependency Graphs

Define:

- $p_i = \Pr[B_i]$
- $\mu = \sum_i p_i$
- $\Theta_1 = \sum_i \sum_{j \in N(i)} p_i p_j$
- $\Theta_2 = \sum_i \sum_{j \in N(i), j \neq i} p_{ij}$ where $p_{ij} = \Pr[B_i \wedge B_j]$
- (If events are not strictly independent outside neighborhood, we could define at Θ_3 measuring this)

Total Variation Distance

Before we state the theorem, we need a definition.

Definition

The Total Variation Distance between two probability measures P and Q on the same $(\Omega, \mathcal{F})$ is defined to be

$$||P - Q||_{TV} = \sup_{A \in \mathcal{F}} |P(A) - Q(A)|$$

Fow two discrete probability measures, this is equivalent to:

$$||P - Q||_{TV} = \frac{1}{2} \sum_{x \in \Omega} |P(x) - Q(x)|$$

Chen-Stein Poisson Approximation

Theorem

For a set of events B_i, with dependency graph and μ, Θ_1, Θ_2 defined as above, let $Z \sim Pois(\mu)$. Then

$$||X - Z||_{TV} \leq 2(\Theta_1 + \Theta_2)$$