Probability Spaces

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Sigma Fields

Definition

A sigma-field (σ-field) $\mathcal{F}$ is a collection (family) of subsets of a space Ω satisfying:

1. $\Omega \in \mathcal{F}$
2. *If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$*
3. *If $A_1, A_2, \cdots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$*

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- $\mathcal{F} = \{\Omega, \emptyset\}$
- $\mathcal{F} = \{\Omega, \emptyset, A, A^c\}$

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Let Ω be a metric space and $\mathcal{F}$ the collection of all open subsets of Ω. Then the Borel σ-field is $\sigma(\mathcal{F})$

Definition

Let $\mathcal{F}$ be a σ-field on Ω. Then $P : \mathcal{F} \to [0, 1]$ is a probability measure if:

1. $P(\Omega) = 1$
2. For any $A_1, A_2, \cdots \in \mathcal{F}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Probability Space

Definition

A *probability space* is a triple $(\Omega, \mathcal{F}, P)$ of a space, a σ-field, and a probability function.

Examples

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2. Two coin flips: $\Omega = \{HH, HT, TH, TT\}$, $\mathcal{F} = \{$ all subsets $\}$, $P(HH) = 1/4, \ldots$
3. Pick a uniform random number from $[0, 1]$: $\Omega = [0, 1]$. $\mathcal{F}$ is the Lebesgue or Borel σ-field. P is Lebesgue measure (i.e., $P([a, b]) = b - a$, the length of the interval).

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5. Complement: A^c is the event that A does not happen.

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$$P(A) = \lim_{n \to \infty} P(A_n)$$
6. For any $B \in \mathcal{F}$, $P(A) = P(A \cap B) + P(A \cap B^c)$