Random Graphs

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Graph Terminology

A graph $G = (V, E)$ is a set of vertices and a set of pairs of vertices called edges.

There are many different ways of drawing the same graph.

Graph Terminology

Some special graphs:

1. The complete graph on n vertices, K_n. All edges present.
2. A cycle on n vertices C_n
3. A bipartite graph: all edges cross a partition.

Graph Terminology

Some terms to know:

- Clique: a set of vertices each of which is joined to the rest. Eg. a triangle is a 3-clique.
- Isolate vertex
- Path
- Connected Graph
- Tree

Erdős-Rényi

The Erdős-Rényi random graph comes in two varieties, one proposed by the two Hungarians and one proposed by Edward Gilbert, both in 1959.

$G(n, m)$ is a graph on n vertices chosen uniformly at random from the set of all graphs with exactly m edges.

$G(n, p)$ is a graph on n vertices in which each of the $\binom{n}{2}$ potential edges is present with probability p.

Thresholds

A *graph property* is a collection of graphs closed under permutations of the vertices.

Definition

A monotone increasing proprty is a property $\mathcal{P}$ so that if $G \in \mathcal{P}$, then $G + \{e\} \in \mathcal{P}$ for every edge e.

Thresholds

We often are interested in thresholds for monotone properties in random graphs. In the $G(n,p)$ model

Definition

p^* is a threshold for $\mathcal{P}$ if

1. for $p >> p^*$, $\Pr[G(n,p) \in \mathcal{P}] \to 1$
2. for $p << p^*$, $\Pr[G(n,p) \in \mathcal{P}] \to 0$

Examples

1. Show that $p = 1/n$ is a threshold for the appearance of a triangle in the random graph.
2. Show that $p = \log n/n$ is a threshold for the disappearance of isolated vertices.
3. How are these thresholds different?

Sharp Thresholds

Definition

p^* is a sharp threshold for $\mathcal{P}$ if for every $\epsilon > 0$,

1. for $p > (1+\epsilon)p^*$, $\Pr[G(n,p) \in \mathcal{P}] \to 1$
2. for $p < (1-\epsilon)p^*$, $\Pr[G(n,p) \in \mathcal{P}] \to 0$

Which of the previous properties has a sharp threshold?