Random Walks

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Simple Random Walk

$S_0 = 0$,
$S_n = X_1 + X_2 + \ldots X_n$, where X_i's are iid ± 1 with probability p and $1 - p$

Combinatorics

Calculating the number of simple random walk paths from $(0, 0)$ to (n, k) is finding a binomial coefficient, the only trick is figuring out the number of 'up steps'.

$$U + D = n$$
$$U - D = k$$
$$U = \frac{n+k}{2}$$

Probabilities

To find $\Pr[S_n = k]$ multiply the number of walks by the probability of each walk:

$$\Pr[S_n = k] = \binom{n}{\frac{n+k}{2}} p^{\frac{n+k}{2}} (1-p)^{n-\frac{n+k}{2}}$$

Example

Find the asymptotics of $\Pr[S_n = \alpha n]$ for a p-biased SRW.
[I.e., find a function $g(n)$ so that $\Pr[S_n = \alpha n] \sim g(n)$.]

Exact:

$$\Pr[S_n = \alpha n] = \binom{n}{\frac{(1+\alpha)n}{2}} p^{\frac{(1+\alpha)n}{2}} (1-p)^{\frac{(1-\alpha)n}{2}}$$

Asymptotics? Use Stirling's Formula and Logs
When is this probability exponentially small in n? When is it polynomially small?

Random Walk in Higher Dimension

We can define a simple random walk in $2d$, $3d$, or higher. In $2d$ the walk starts at $(0, 0)$ and takes steps on the integer lattice. The set of possible moves are the 4 neighbors of the current location: for $(0, 0)$ these are $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.
Again there is a parity issue: one even steps, the sum of all coordinates is even, and on odd steps, the sum is odd.

Random Walk in Higher Dimension

Find the asymptotics of $\Pr[S_n = (0, \ldots, 0)]$ for a SSRW in d dimensions.

Random Walk in Higher Dimension

$d = 2$: Idea: project the walk to the two diagonal lines $y = x$ and $y = -x$, i.e. use a change of basis.

In terms of these new coordinates, what happens with one step of the walk?

Random Walk in Higher Dimension

What about for $d = 3$? Can't just change coordinates.

Give an upper and a lower bound on $\Pr[S_n^{(3)} = (0, 0, 0)]$

Reflection Principle

What is the number of simple random walk paths that go from $(0, 0)$ to $(20, 10)$ without going below the x-axis?

Draw a picture!

Ballot Theorem

In an election, candidate A gets A votes, beating candidate B with B votes.

If the votes are counted in a random order, what's the probability candidate A will always be ahead in the count?

Ballot Theorem

Theorem

If candidate A receives A votes and B receives B, $A > B$, then the probability A is always ahead in the counting is

$$\frac{A-B}{A+B}$$

Proof by reflection principle