Stochastic Processes

Will Perkins

March 7, 2013

Stochastic Processes

Q: What is a Stochastic Process?

A: A collection of random variables defined on the same probability space and indexed by a 'time' parameter.

$$\{Z_t\}_{t \in \mathcal{T}}$$

where each $Z_t \in \mathcal{X} \subseteq \mathbb{R}$.

Example: a Simple Random walk is the collection $\{S_n\}_{n \in \mathbb{Z}_+}$

Another viewpoint: a stochastic process is a random *function* from $\mathcal{T} \to \mathcal{X}$.

Types of Processes

There are 4 broad types of stochastic processes:

1. Discrete time, discrete space: $\mathcal{T} = \mathbb{Z}_+$, $\mathcal{X}$ countable. Eg. simple random walk, Galton-Watson branching process.
2. Discrete time, continuous space: $\mathcal{T} = \mathbb{Z}_+$, $\mathcal{X} = \mathbb{R}$. Eg. a random walk whose steps have a Normal distribution.
3. Continuous time, discrete space: $\mathcal{T} = \mathbb{R}_+$, $\mathcal{X}$ countable. Eg. a 'Jump' process. Queuing models, i.e. X_t is the number of people in line at a bank at time t.
4. Continuous time, continuous space: $\mathcal{T} = \mathbb{R}_+$, $\mathcal{X} = \mathbb{R}$. Eg. Brownian Motion.

For now we will consider discrete time, discrete space processes. We will often index our state space by integers since it is countable.

Markov Processes

Definition

A stochastic process S_n is a Markov Chain if

$$\Pr[S_n = x | S_0 = x_0, S_1 = x_1, \ldots S_{n-1} = x_{n-1}] = \Pr[S_n = x | S_{n-1} = x_{n-1}]$$

for all choices of $x, x_1, \ldots x_{n-1}$.

Exercise 1: Prove that a simple random walk is a Markov Chain.

Exercise 2: Find an example of a random process that is not a Markov Chain.

Transition Probabilities

For a markov chain, the probability of moving from state i to state j at step n depends only on 3 things: i, j, and n. The transition probabilities are the collection of probabilities

$$p_{i,j}(n) = \Pr[S_n = j | S_{n-1} = i]$$

What are the transition probabilities for a simple random walk?

Homogeneous Markov Chains

SRW is an example of a class of particularly simple Markov Chains:

Definition

A Markov Chain is called homogeneous if

$$p_{i,j}(n) = p_{i,j}(m)$$

for all i, j, n, m.

In this case we simply write $p_{i,j}$.

Transition Matrix

The transition matrix of a homogeneous Markov Chain is the $|\mathcal{X}| \times |\mathcal{X}|$ matrix P with entries

$$P_{ij} = p_{i,j}$$

Properties of a transition matrix:

1. $P_{ij} \geq 0$ for all i, j
2. $\sum_j P_{ij} = 1$ for all j.

Such matrices are also called Stochastic Matrices.

Chapman-Kolmogorov Equations

Let $p_{i,j}(n, m) = \Pr[S_m = j | S_n = i]$.

Theorem (Chapman-Kolmogorov Equations)

$$p_{i,j}(n, n+m+r) = \sum_{k \in \mathcal{X}} p_{i,k}(n, n+m) p_{k,j}(n+m, n+m+r)$$

for all choices of the parameters.

Some Linear Algebra

Define a matrix P_n with the i, jth entry being $\Pr[S_n = j | S_0 = i]$. Then

$$P_n = P^n$$

Proof: Use Chapman Kolmogorov Equations.

Distribution of the Chain

One thing we would like to know about a Markov Chain is where it is likely to be at some step n. We can keep track of this with a vector of length $|\mathcal{X}|$, $\mu^{(n)}$, where

$$\mu_i^{(n)} = \Pr[S_n = i]$$

Given $\mu^{(0)}$, what is $\mu^{(1)}$?

$$\mu^{(1)} = \mu^{(0)} P$$

[Check this for SRW]

In general,

$$\mu^{(n)} = \mu^{(0)} P^n$$

Transience and Recurrence

Definition

A state $x \in \mathcal{X}$ is recurrent if
$\Pr[S_n = x \text{ for some } n \geq 1 | S_0 = x] = 1$.

Definition

A state x is called transient if it is not recurrent.

Transience and Recurrence

Is SRW recurrent or transient?
We will prove a general theorem that will allow us to determine this for SRW in any dimension.

Step 1: Define the hitting probabilities:

$$f_{ij}(n) = \Pr[S_1 \neq j, \dots S_{n-1} \neq j, S_n = j | S_0 = i]$$

Let

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$$

A state i is recurrent if and only if $f_{ii} = 1$.

Transience and Recurrence

Step 2: Define 2 generating functions:

$$P_{ij}(s) = \sum_{n=0}^{\infty} s^n p_{ij}(n)$$

and

$$F_{ij}(s) = \sum_{n=0}^{\infty} s^n f_{ij}(n)$$

We assume $p_{ij}(0) = 1$ iff $i = j$ and $f_{ij}(0) = 0$ for all i, j. Fact: $F_{ij}(1) = f_{ij}$.

Transience and Recurrence

Step 3:

Lemma

- $P_{ii}(s) = 1 + F_{ii}(s)P_{ii}(s)$
- $P_{ij}(s) = F_{ij}(s)P_{jj}(s)$ *if* $i \neq j$.

Proof:

Step 4:

Corollary

State i is recurrent if and only if

$$\sum_n p_{ii}(n) = \infty$$

Proof:

Positive and Null Recurrent

Definition

The mean recurrence time of a state i, $\mu(i)$, is the expected number of steps required to return to state i after starting at state i.

$$\mu(i) = \sum_{n=1}^{\infty} n f_{ii}(n)$$

if i is recurrent and $\mu(i) = \infty$ if i is transient.

Definition

Let i be a recurrent state. If $\mu(i) = \infty$ then we call i null recurrent. If $\mu(i) < \infty$, then i is called positive recurrent.

Positive and Null Recurrent

Lemma

A recurrent state i is null recurrent if and only if

$$p_{ii}(n) \to 0 \text{ as } n \to \infty$$