

ANALYTICITY FOR CLASSICAL GASSES VIA RECURSION

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ABSTRACT. We give a new criterion for a classical gas with a repulsive pair potential to exhibit uniqueness of the infinite volume Gibbs measure and analyticity of the pressure. Our improvement on the bound for analyticity is by a factor e^2 over the classical cluster expansion approach and a factor e over the known limit of cluster expansion convergence. The criterion is based on a contractive property of a recursive computation of the density of a point process. The key ingredients in our proofs include an integral identity for the density of a Gibbs point process and an adaptation of the algorithmic correlation decay method from theoretical computer science. We also deduce from our results an improved bound for analyticity of the pressure as a function of the density.

1. INTRODUCTION

A central goal in classical statistical mechanics is to derive macroscopic properties of gasses, liquids, and solids given only details of microscopic interactions. The classical model involves indistinguishable particles interacting in the continuum via a many-body potential. Even in the case of a pair potential fundamental questions remain open, such as whether or not such a model exhibits a phase transition from a gaseous to a solid state. One major success of the field is in the study of convergent expansions for thermodynamic quantities in powers of the activity or density; using these techniques, the model is proved to be in a gaseous state throughout the region of convergence of these series. However, convergence can be limited by the presence of non-physical singularities in the complex plane. Here, in the case of repulsive pair potentials, we extend the known range of the gaseous phase using analytic techniques that avoid such singularities and focus on small complex neighborhoods of the positive real axis.

1.1. **The model.** We consider classical particles interacting in a finite volume $\Lambda \subset \mathbb{R}^d$ via a symmetric, translation invariant pair potential ϕ .

The energy of a configuration $\{x_1, \dots, x_n\} \subset \mathbb{R}^d$ is given by

$$U(x_1, \dots, x_n) = \sum_{1 \leq i < j \leq n} \phi(x_i - x_j).$$

We make the following assumptions on the potential $\phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$:

- (1) ϕ is *repulsive*: $\phi(x) \geq 0$ for all x .
- (2) ϕ is *tempered*: $\int_{\mathbb{R}^d} |1 - e^{-\phi(x)}| dx < \infty$.

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The grand canonical partition function at activity $\lambda \geq 0$ on a bounded region $\Lambda \subset \mathbb{R}^d$ is given by:

$$(1) \quad Z_\Lambda(\lambda) = 1 + \sum_{k \geq 1} \frac{\lambda^k}{k!} \int_\Lambda \cdots \int_\Lambda e^{-U(x_1, \dots, x_k)} dx_1 \cdots dx_k.$$

(Without losing any generality, we absorb the usual inverse temperature parameter β into the potential ϕ).

If we let Λ_V be the axis-parallel box of volume V centered at the origin in \mathbb{R}^d , then the infinite volume pressure is

$$p_\phi(\lambda) = \lim_{V \rightarrow \infty} \frac{1}{V} \log Z_{\Lambda_V}(\lambda).$$

See e.g. [34] for a proof of the existence of the limit. Non-analytic points of $p_\phi(\lambda)$ on the positive real axis mark phase transitions of the infinite volume system. An important topic in classical statistical physics is proving the absence of phase transitions for certain parameter values; that is, proving analyticity of the pressure. We remark that proving the existence of a phase transition in the models considered here is a notoriously challenging problem; to the best of our knowledge it is not proved for any monatomic gas interacting through a finite-range or rapidly decaying pair potential (see [42, 21] for proofs of phase transitions in continuous particle systems with different types of interactions). For more background on classical gasses see [34].

1.2. Main results. Our main result is a new criterion for analyticity of the pressure and uniqueness of the infinite volume Gibbs measure.

Theorem 1. *Let ϕ be a repulsive, tempered potential and let*

$$(2) \quad C_\phi = \int_{\mathbb{R}^d} 1 - e^{-\phi(x)} dx.$$

Then for $\lambda \in [0, e/C_\phi)$ the infinite volume pressure $p_\phi(\lambda)$ is analytic and there is a unique infinite volume Gibbs measure.

This improves by a factor e^2 the classical bound of $1/(eC_\phi)$ obtained by Groeneveld in 1962 using the cluster expansion [14]. Extensions by Penrose [28] and Ruelle [33] a year later to a wider class of potentials via the Kirkwood–Salsberg equations match Groeneveld’s bound in the case of repulsive potentials. In a remarkable but little noticed paper, Meeron [26] proved uniqueness and analyticity for $\lambda < 1/C_\phi$ using a novel interpolation scheme [25] and recurrence relations for k -point densities related to the Kirkwood–Salsberg equations.

New criteria for cluster expansion convergence have been given by Faris [9], Jansen [18], and Nguyen and Fernández [27] based on the work of Fernández and Procacci in discrete setting [10] and the extension by Fernández, Procacci, and Scoppola to hard spheres [11]. Explicit improvements to the classical bounds from the new criteria have been worked out in the case of hard spheres in dimension 2 (see Section 1.2.1 below).

Moreover, $1/C_\phi$ is an upper bound on the radius of convergence of the cluster expansion for repulsive potentials [14, 28] (see remark 3.7 in [27]). Theorem 1 surpasses this limit (and Meeron’s bound) by a factor e .

Using Theorem 1 we can also deduce that the pressure is analytic with respect to the infinite volume density defined by

$$\rho_\phi(\lambda) := \lambda \frac{d}{d\lambda} p_\phi(\lambda).$$

Theorem 2. *For any repulsive, tempered potential ϕ , the infinite volume pressure p_ϕ is analytic as a function of the density ρ_ϕ for $\rho_\phi \in [0, \frac{e}{1+e} \frac{1}{C_\phi}]$.*

Previous results showed analyticity of the pressure as a function of the density by showing convergence of the virial series in a disk around 0 in the complex plane [22, 30, 19, 27]. The best bound on convergence of the virial expansion for general repulsive potentials is that proposed by Groeneveld [15] and proven by Ramawadh and Tate [31], showing convergence for $|\rho_\phi| \leq .237961 \frac{1}{C_\phi}$.

1.2.1. *Example: the hard sphere model.* One of the most studied classical gasses is the hard sphere model, with the potential $\phi(x) = +\infty$ if $\|x\| < r$ and 0 otherwise. A configuration (x_1, \dots, x_k) with $U(x_1, \dots, x_k) = 0$ represents the centers of a packing of k spheres of radius $r/2$. This model provides a good testing ground for different criteria for uniqueness and analyticity.

For convenience, let us take r to be the radius of the ball of volume 1 in \mathbb{R}^d . Then the classical Groeneveld–Penrose–Ruelle bound gives convergence of the cluster expansion and uniqueness for $\lambda < 1/e \approx .3679$. Fernández, Procacci, and Scoppola [11] gave an improved bound in terms of multi-dimensional integrals. For $d = 2$ their improved bound is $\lambda < .5017$.

Hofer-Temmel [17] and Dereudre [6] gave a new bound for uniqueness and exponential decay of correlations for the hard sphere model based on disagreement percolation and the critical activity for Poisson-Boolean percolation. In high dimensions this gives uniqueness for $\lambda < 1 + o_d(1)$. In dimension 2, using the ‘high-confidence’ bound for Poisson-Boolean percolation from [1], this approach gives uniqueness for $\lambda < 1.127$, a significant improvement over the cluster expansion approach. This approach also works for repulsive potentials with finite range [6, 4] As discussed in [18], however, the advantage of the disagreement percolation method over the cluster expansion method is greatest for hard spheres: for general repulsive potentials the cluster expansion can do better since the disagreement percolation bound is just in terms of the range r of the potential.

Recently, Helmuth, Petti, and the second author [16] improved these bounds in all dimensions by showing uniqueness and exponential decay of correlations in the hard sphere model for $\lambda < 2$ using path coupling to show rapid mixing of a discrete-time Markov chain.

Theorem 1 implies a further improvement to these bounds.

Corollary 3. *Consider the hard sphere model with spheres of radius $r/2$ where r is the radius of the ball of volume 1 in \mathbb{R}^d . Then for $\lambda < e$ the pressure is analytic and there is a unique infinite-volume Gibbs measure.*

Theorem 2 also gives an improvement on the bounds for analyticity of the pressure as a function of the density of the hard sphere model. We write these bounds in terms of the packing density of the model, which given our choice of normalization is $2^{-d} \rho_\phi(\lambda)$ since the volume of the ball of radius $r/2$ is 2^{-d} .

Corollary 4. *The infinite volume pressure of the hard sphere model in dimension d is analytic as a function of the density for packing densities at most $\frac{e}{1+e} 2^{-d}$. Moreover, for any $\varepsilon > 0$*

and d large enough, the pressure of the hard sphere model in dimension d is analytic for packing densities at most $(1 - \varepsilon)2^{-d}$.

For instance, in dimension 2 this gives analyticity up to packing density .18276. The best previous lower bound on the critical density (in terms of uniqueness of the infinite volume Gibbs measure) was $1/6$ from [16]. The best bound obtained via convergence of the virial expansion is .0751 by Nguyen and Fernández [27]. The new bound is still far from the predicted critical packing density near .7 [5, 8].

Packing density 2^{-d} is a natural barrier to analysis since below 2^{-d} free volume (space in which new centers can be placed) is guaranteed by a union bound. Improving the asymptotic bound in Corollary 4 by any constant factor would likely require significant new insight.

1.3. Absence of zeros in the complex plane. To prove Theorem 1 we will work in the setting of a multivariate, complex-valued partition function. Given any bounded, measurable function $\lambda : \Lambda \rightarrow \mathbb{C}$, we define

$$(3) \quad Z_\Lambda(\lambda) = \sum_{k \geq 0} \frac{1}{k!} \int_{\Lambda^k} \prod_{i=1}^k \lambda(x_i) \cdot e^{-U(x_1, \dots, x_k)} dx_1 \cdots dx_k.$$

When λ is constant this definition coincides with (1).

Following the Lee-Yang theory of phase transitions [43], absence of phase transitions and analyticity of the pressure for activities in $[0, \lambda_0]$ is closely related to the existence of a region R in the complex plane containing the segment $[0, \lambda_0]$ so that for any $\lambda \in R$ and any bounded region Λ , $Z_\Lambda(\lambda) \neq 0$. Theorem 1 thus follows, after a little complex analysis in Section 5, from the next theorem.

Theorem 5. *Let ϕ be a repulsive, tempered potential and suppose $\lambda \in (0, e/C_\phi)$. Then there exist $\varepsilon > 0$ and $C > 0$, so that the following holds. Let L_0 be the ε -neighborhood of the interval $[0, \lambda]$ in the complex plane. Then for any measurable function $\lambda : \mathbb{R}^d \rightarrow \mathbb{C}$ so that $\lambda(x) \in L_0$ for all x , and any bounded, measurable $\Lambda \subset \mathbb{R}^d$, we have*

$$|\log Z_\Lambda(\lambda)| \leq C|\Lambda|.$$

In particular, $Z_\Lambda(\lambda) \neq 0$.

1.4. Outline of techniques. The classical approach to proving absence of phase transitions, convergence of the cluster expansion, is limited by the possible presence of a complex zero of the partition function far from the positive real axis which determines the radius of convergence of the cluster expansion but does not affect the physical behavior of the system. In fact, for repulsive gasses, the closest singularity of the pressure to the origin lies on the negative real axis [14]. Therefore to move beyond the limits of cluster expansion convergence one must use properties of positive activities or utilize regions of the complex plane that are not symmetric around 0. Two previous approaches in this direction are probabilistic: disagreement percolation [6, 17, 4] and Markov chain mixing [16]. In the case of the hard sphere model these techniques surpass the bounds for uniqueness given by the cluster expansion. One drawback is that these arguments rely on a finite-range property of a potential (like hard spheres) and it is not clear how to extend these arguments to potentials satisfying the more natural temperedness assumption.

Our approach to proving Theorem 1 will instead be analytic, using essentially no probabilistic tools. One consequence of the convergence of the cluster expansion is that $Z_\Lambda(\lambda)$

is not zero for λ in a disk in the complex plane, uniformly in the region Λ . To avoid the singularity on the negative real axis, we instead prove that $Z_\Lambda(\lambda)$ is not zero in an asymmetric region in the complex plane that contains $[0, e/C_\phi)$. We obtain this zero-free region by proving recursive bounds on the log partition function and on a generalization of the density of a point process to complex parameters.

Our approach is inspired by the ‘correlation decay method’, an algorithmic technique from theoretical computer science for approximating the partition function of a discrete spin model. The method was introduced by Weitz [40] in an influential paper on approximate counting and sampling from the hard core model on a graph G . At the core of Weitz’s argument is a recursion for the marginal probability that a given vertex v is chosen in the independent set in terms of the marginal probabilities of its neighbors in a modified model that removes the dependence between neighbors of v . This ultimately allows Weitz to construct a ‘self-avoiding walk tree’ T with root r so that the marginal of v in G equals the marginal of r in T . A monotonicity argument then shows that if the infinite Δ -regular tree exhibits strong spatial mixing at activity λ then every graph of maximum degree Δ does as well. The threshold for strong spatial mixing on the Δ -regular tree can be explicitly determined via fixed point equations as $\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^\Delta}$. Consequences include an efficient algorithm for approximating the partition function on any graph of maximum degree Δ for $\lambda < \lambda_c(\Delta)$, and a lower bound on the uniqueness threshold of the hard-core model on \mathbb{Z}^d of $\lambda_c(2d)$. The correlation decay method has since been refined and used to prove the best current lower bounds on the uniqueness threshold for the hard-core lattice gas model on \mathbb{Z}^2 [39, 37].

Most relevant for our approach is the paper of Peters and Regts [29] in which they use a recursion inspired by Weitz and ideas from complex dynamical systems to prove the existence of a zero-free region for the independence polynomial of graphs of maximum degree Δ in a complex neighborhood containing $[0, \lambda_c(\Delta))$, thus solving a conjecture of Sokal [38]. Their work was in part motivated by another approach to approximate counting, the polynomial interpolation method of Barvinok [3].

Our goal in this paper is to develop a version of the correlation decay method for continuous particle systems. This presents several challenges including determining the right analogue of self-avoiding walk trees and recurrence relations for point processes.

We work with a generalization of the definition of the density of a point process to complex-valued activities (given in (4) below and used already in [33]). We develop three tools to work with these densities. The first is Lemma 7, an integral expression for $\log Z_\Lambda(\lambda)$ in terms of these generalized densities. This is what allows us to bound $\log Z_\Lambda(\lambda)$ and prove the absence of zeroes in a region in the complex plane. The next is Theorem 8 which gives a recursive integral identity for complex densities. This is somewhat analogous to one step of the self-avoiding walk tree construction of Godsil [13] and Weitz [40] in discrete settings. Finally our convergence criteria is determined by the contractive properties of the functional that defines this recursive identity given in Lemma 12.

At a very high level, our approach has some similarity to the approach of Penrose and Ruelle via the Kirkwood-Salsberg equations and that of Meeron [26]: we write identities involving densities and show that for a certain range of complex parameters the operator defining these identities is contractive in a suitable sense. In Ruelle’s argument, uniqueness follows from invertibility of $1 - \lambda K$ where K is an operator on a Banach space. When the norm of λK is strictly less than 1 then invertibility, and thus uniqueness, follows. Without a deeper understanding of the spectrum of this operator K , this approach inherently only

provides uniqueness for λ in a disk centered at 0. Conversely, we work entirely with values of λ near the positive interval $[0, e/C_\phi)$ which allows us to avoid the non-physical obstructions to analyticity on the negative real axis. Meeron's interpolation between zero interaction and full interaction also avoids the singularity on the negative real axis, but the operator defining his recursion is contractive only for $\lambda < 1/C_\phi$; it would be interesting to see if combining his approach with a change of coordinates as in Section 3 below could match the bound of Theorem 1. One could view successive applications of Theorem 8 below as a non-uniform interpolation from activity 0 to activity λ .

1.5. Future directions. While we work here only with repulsive potentials, we would be very interested to see if the results could be generalized to the class of stable, tempered potentials, the setting of the results of Penrose [28] and Ruelle [33] which include more physically relevant examples such as the Lennard–Jones potential. For a discussion from the physics perspective of the utility of purely repulsive potentials such as the hard sphere mode, see [41, 2].

When specialized to real parameters, the functional contraction properties given in Section 3 can be used to show that a recursive computation of the density exhibits exponentially small dependence on boundary conditions in the height of the recursion. This in turn can be used to show strong spatial mixing for finite-range, repulsive potentials. In light of these properties and the algorithmic roots of our techniques, it would be interesting to use these methods to design efficient algorithms for estimating the pressure or density when $\lambda < e/C_\phi$.

We believe that one could take advantage of the geometry of low dimensional Euclidean space (e.g. $d = 2, 3$) and an appropriate notion of the connective constant of \mathbb{R}^d to improve the bounds of Theorem 1, as was done for the hard-core lattice gas by Sinclair, Srivastava, Štefankovič, and Yin [37]. Doing this for the hard sphere model in dimension 2 would test the limits of this method and the analogy with discrete spin systems.

1.6. Notation. A *region* is a bounded, measurable subset of \mathbb{R}^d . The volume of $\Lambda \subset \mathbb{R}^d$ is denoted $|\Lambda|$. We denote by $\text{dist}(x, y)$ the Euclidean distance between $x, y \in \mathbb{R}^d$. An *activity function* on a region Λ is a bounded, measurable function from $\Lambda \rightarrow \mathbb{C}$. We use a bold symbol, e.g. $\boldsymbol{\lambda}$, for a non-constant activity function. Throughout, a *complex neighborhood* is a bounded open subset of \mathbb{C} ; in each case here, the complex neighborhoods in question are conformal images of the open unit disk, and thus are simply connected as well. For positive parameters λ and ε , we write $\mathcal{N}(\lambda, \varepsilon)$ to be the complex neighborhood $\{z \in \mathbb{C} : \text{dist}(z, [0, \lambda]) < \varepsilon\}$. For a set A we write $\mathbf{1}_A$ to be the indicator function of the set A .

2. GENERALIZED DENSITIES

To prove Theorem 5, we use a generalization of the density of a point process to complex activity functions: the density of $v \in \Lambda$ at activity $\boldsymbol{\lambda}$ is

$$(4) \quad \rho_{\Lambda, \boldsymbol{\lambda}}(v) = \boldsymbol{\lambda}(v) \frac{Z_{\Lambda}(\boldsymbol{\lambda} e^{-\phi(v-\cdot)})}{Z_{\Lambda}(\boldsymbol{\lambda})}$$

if $Z_{\Lambda}(\boldsymbol{\lambda}) \neq 0$. If $Z_{\Lambda}(\boldsymbol{\lambda}) = 0$ then the density is undefined. Here $\boldsymbol{\lambda} e^{-\phi(v-\cdot)} : \Lambda \rightarrow \mathbb{C}$ denotes the function $\boldsymbol{\lambda}(x) e^{-\phi(v-x)}$.

In Section 5 we will also work with multipoint densities: the k -point density (or k -point correlation function) of $(v_1, \dots, v_k) \in \Lambda^k$ at activity λ is

$$(5) \quad \rho_{\Lambda, \lambda}(v_1, \dots, v_k) = \lambda(v_1) \cdots \lambda(v_k) \frac{Z_{\Lambda}(\lambda e^{-\sum_{i=1}^k \phi(v_i - \cdot)})}{Z_{\Lambda}(\lambda)} e^{-U(v_1, \dots, v_k)}.$$

Remark. There is a natural probabilistic interpretation of the densities when λ is non-negative. First define the *Gibbs point process* \mathbf{X} to be the random point set in Λ with density $e^{-U(\cdot)}$ against the Poisson process with intensity λ . Then the density $\rho_{\Lambda, \lambda}$ is the density of the measure that assigns to a set A the mass $\mathbb{E}[|\mathbf{X} \cap A|]$ with respect to Lebesgue measure; a similar fact holds for the multipoint density with $\mathbb{E}|\mathbf{X} \cap A|$ replaced by a certain factorial moment. When λ is non-negative, these may be taken as a definition of the density $\rho_{\Lambda, \lambda}$ and (4) and (5) become identities to check (see, e.g. [34, Chapter 4]). Since we are interested in complex values of λ , we use (4) and (5) as our definition. This idea appears in Ruelle's classic text on statistical mechanics [34, Chapter 4] as well, in which this identity appears in a slightly different form to extend the definition of $\rho_{\Lambda, \lambda}$ to complex λ . Where our definition (4) differs from Ruelle, however, is the use of a "multivariate" λ rather than constant λ which allows us to write $\rho_{\Lambda, \lambda}$ as λ times a ratio of partition functions. This turns out to be a crucial feature of (4) and essentially allow us to prove zero-freeness of $Z_{\Lambda}(\lambda)$ inductively (see Theorem 19).

In what follows in Sections 2, 3, and 4 the region Λ will be fixed, and so we will drop the subscript Λ from the notation, writing $Z(\lambda)$ for $Z_{\Lambda}(\lambda)$ and $\rho_{\lambda}(x)$ for $\rho_{\Lambda, \lambda}(x)$. We will interpret an activity function λ on Λ as a function $\lambda : \mathbb{R}^d \rightarrow \mathbb{C}$ that is 0 on $\mathbb{R}^d \setminus \Lambda$. In fact, Lemma 7 and Theorem 8 below hold for bounded, integrable functions $\lambda : \mathbb{R}^d \rightarrow \mathbb{C}$ without requiring bounded support. We call such functions *activity functions*.

Given an activity function λ , let $\mathcal{A}_{\lambda} = \{\lambda' = \alpha \cdot \lambda : \alpha : \mathbb{R}^d \rightarrow [0, 1] \text{ is measurable}\}$. We will use the following hereditary notion of zero freeness.

Definition 6. *An activity function λ is totally zero-free if $Z(\lambda') \neq 0$ for all $\lambda' \in \mathcal{A}_{\lambda}$.*

Our first lemma is an integral identity for the log partition function.

Lemma 7. *If an activity function λ is totally zero-free then*

$$\log Z(\lambda) = \int_{\mathbb{R}^d} \rho_{\hat{\lambda}_x}(x) dx$$

where

$$\hat{\lambda}_x(y) = \begin{cases} 0 & \text{if } y \in \Lambda_x \\ \lambda(y) & \text{if } y \notin \Lambda_x \end{cases}$$

and $\Lambda_x = \{y \in \mathbb{R}^d : \text{dist}(0, y) < \text{dist}(0, x)\}$.

Proof. Define $\lambda_t(y) = \mathbf{1}_{\text{dist}(y, 0) \geq t} \lambda(y)$ and note by assumption $Z(\lambda_t) \neq 0$ for all t . We will apply the fundamental theorem of calculus and integrate $\frac{d}{dt} \log Z(\lambda_t)$. Since we want to compute $\frac{d}{dt} \log Z(\lambda_t)$, we compute $\frac{d}{dt} Z(\lambda_t)$; first change coordinates to write

$$Z(\lambda_t) - 1 = \sum_{j \geq 1} \frac{1}{j!} \int_{(\mathbb{R}^d)^j} \prod_{i=1}^j \lambda_t(v_i) e^{-U(v_1, \dots, v_j)} d\mathbf{v}$$

$$\begin{aligned}
&= \sum_{j \geq 1} \frac{1}{j!} \int_0^\infty \int_{\partial(B_s(0)^j)} \prod_{i=1}^j \lambda_t(v_i) e^{-U(v_1, \dots, v_j)} d\mathbf{v} ds \\
&= \sum_{j \geq 1} \frac{1}{j!} \int_0^\infty j \int_{\partial B_s(0)} \lambda_t(w) \int_{B_s(0)^{j-1}} \prod_{i=1}^{j-1} \lambda_t(v_i) e^{-U(v_1, \dots, v_{j-1}, w)} d\mathbf{v} dw ds \\
&= \sum_{j \geq 1} \frac{1}{j!} \int_t^\infty j \int_{\partial B_s(0)} \lambda(w) \int_{B_s(0)^{j-1}} \prod_{i=1}^{j-1} \lambda_t(v_i) e^{-U(v_1, \dots, v_{j-1}, w)} d\mathbf{v} dw ds
\end{aligned}$$

and so we have

$$\frac{d}{dt} Z(\lambda_t) = - \sum_{j \geq 1} \frac{1}{j!} \cdot j \int_{\partial B_t(0)} \lambda(w) \int_{(\mathbb{R}^d)^{j-1}} \prod_{i=1}^{j-1} \lambda_t(v_i) e^{-U(v_1, \dots, v_{j-1}, w)} d\mathbf{v} dw$$

where the outermost integral is a $d - 1$ dimensional integral over the boundary of the ball of radius t centered at 0. Rearranging the terms, we have

$$\begin{aligned}
\frac{d}{dt} Z(\lambda_t) &= - \int_{\partial B_t(0)} \lambda(w) \sum_{j \geq 0} \frac{1}{j!} \int_{(\mathbb{R}^d)^j} \prod_{i=1}^j \left(\lambda_t(v_i) e^{-\phi(v_i - w)} \right) e^{-U(v_1, \dots, v_j)} d\mathbf{v} dw \\
&= - \int_{\partial B_t(0)} \lambda(w) Z(\lambda_t e^{-\phi(w - \cdot)}) dw.
\end{aligned}$$

The fundamental theorem of calculus then gives

$$\begin{aligned}
\log Z(\lambda_\infty) - \log Z(\lambda_0) &= \int_0^\infty \frac{d}{dt} \log(Z(\lambda_t)) dt \\
&= - \int_0^\infty \int_{\partial B_t(0)} \lambda(w) \frac{Z(\lambda_t e^{-\phi(w - \cdot)})}{Z(\lambda_t)} dw dt \\
&= - \int_{\mathbb{R}^d} \rho_{\lambda_w}(w) dw.
\end{aligned}$$

Noting $Z(\lambda_\infty) = 1$ and $Z(\lambda_0) = Z(\lambda)$ completes the Lemma. \square

Next we give a recursive identity for the densities. Fix an activity function λ . For $v, w \in \mathbb{R}^d$ let $\lambda_{v \rightarrow w} : \mathbb{R}^d \rightarrow \mathbb{C}$ be defined by

$$\lambda_{v \rightarrow w}(x) = \begin{cases} \lambda(x) e^{-\phi(v-x)} & \text{if } \text{dist}(v, x) < \text{dist}(v, w) \\ \lambda(x) & \text{if } \text{dist}(v, x) \geq \text{dist}(v, w). \end{cases}$$

In particular, $\lambda_{v \rightarrow w} \in \mathcal{A}_\lambda$.

Theorem 8. *Suppose an activity function λ is totally zero-free. Then for every $v \in \mathbb{R}^d$ we have*

$$(6) \quad \rho_\lambda(v) = \lambda(v) \exp \left(- \int_{\mathbb{R}^d} \rho_{\lambda_{v \rightarrow w}}(w) (1 - e^{-\phi(v-w)}) dw \right).$$

Proof. For v fixed and each $t \geq 0$ define

$$\lambda_t(x) = \begin{cases} \lambda(x) e^{-\phi(v-x)} & \text{if } \text{dist}(v, x) < t \\ \lambda(x) & \text{otherwise} \end{cases}.$$

Note that $\lambda_t \in \mathcal{A}_\lambda$, and so by assumption $Z(\lambda_t) \neq 0$.

It will be helpful to write $\lambda_t(x) = \lambda(x)(1 - \mathbf{1}_{\text{dist}(v,x) \leq t}(1 - e^{-\phi(v-x)})$). As in the proof of Lemma 7, we will apply the fundamental theorem of calculus to $\frac{d}{dt} \log Z(\lambda_t)$. We start by computing $\frac{d}{dt} Z(\lambda_t)$. Write

$$Z(\lambda_t) = \sum_{k \geq 0} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} \prod_{j=1}^k \left(\lambda(x_j)(1 - \mathbf{1}_{\text{dist}(v,x_j) \leq t}(1 - e^{-\phi(v-x_j)}) \right) \exp(-U(x_1, \dots, x_k)) \, d\mathbf{x}.$$

Note that for each term in the product, we have

$$\frac{d}{dt}(1 - \mathbf{1}_{\text{dist}(v,x_j) \leq t}(1 - e^{-\phi(v-x_j)})) = -\delta_{\text{dist}(v,x_j)}(t)(1 - e^{-\phi(v-x_j)})$$

where $\delta_{\text{dist}(v,x_j)}(t)$ is the Dirac delta function; equivalently, change coordinates to write $Z(\lambda_t)$ as in the proof of Lemma 7 and use the fundamental theorem of calculus. By the product rule, this implies

$$\begin{aligned} \frac{d}{dt} Z(\lambda_t) &= - \sum_{k \geq 0} \frac{1}{k!} \int_{(\mathbb{R}^d)^{k-1}} \int_{\partial B_t(v)} \prod_{j=1}^{k-1} \lambda_t(x_j) e^{-U(\mathbf{x})} (1 - e^{-\phi(w-v)}) \cdot k \lambda(w) e^{-\sum_{j=1}^{k-1} \phi(x_j-w)} \, d\mathbf{x} \, dw \\ &= - \int_{\partial B_t(v)} \lambda(w) Z(\lambda_t e^{-\phi(w-\cdot)})(1 - e^{-\phi(w-v)}) \, dw. \end{aligned}$$

By the fundamental theorem of calculus, we then have

$$\begin{aligned} \log(Z(\lambda_\infty)) - \log(Z(\lambda_0)) &= \int_0^\infty \frac{\frac{d}{dt} Z(\lambda_t)}{Z(\lambda_t)} \, dt \\ &= - \int_0^\infty \int_{\partial B_t(v)} \lambda(w) \frac{Z(\lambda_t e^{-\phi(w-\cdot)})}{Z(\lambda_t)} (1 - e^{-\phi(w-v)}) \, dw \, dt \\ &= - \int_{\mathbb{R}^d} \rho_{\lambda_{v \rightarrow w}}(w) (1 - e^{-\phi(v-w)}) \, dw. \end{aligned}$$

Noting $\lambda_\infty = \lambda e^{-\phi(v-\cdot)}$ and $\lambda_0 = \lambda$ and applying (4) completes the proof. \square

Finally we need two continuity statements.

Lemma 9. *For any region Λ and any $M > 0$, the map $\lambda \mapsto Z_\Lambda(\lambda)$ is uniformly continuous in the sup norm on the set of activity functions λ with $|\lambda(x)| \leq M$ for all $x \in \Lambda$.*

To prove this we need the following elementary lemma that appears in [7]; we reproduce the simple proof for completeness:

Lemma 10. *Let $\{z_j\}_{j=1}^n$ and $\{w_j\}_{j=1}^n$ be complex numbers with $|z_j|, |w_j| \leq \theta$ for all j . Then*

$$\left| \prod_{j=1}^n z_j - \prod_{j=1}^n w_j \right| \leq \theta^{n-1} \sum_{j=1}^n |w_j - z_j|.$$

Proof. Calculate

$$\left| \prod_{j=1}^n z_j - \prod_{j=1}^n w_j \right| \leq \left| z_n \prod_{j=1}^{n-1} z_j - z_n \prod_{j=1}^{n-1} w_j \right| + \left| z_n \prod_{j=1}^{n-1} w_j - w_n \prod_{j=1}^{n-1} w_j \right|$$

$$\leq \theta \left| \prod_{j=1}^{n-1} z_j - \prod_{j=1}^{n-1} w_j \right| + \theta^{n-1} |z_n - w_n|$$

and induct on n . □

Now we prove Lemma 9.

Proof of Lemma 9. Suppose λ, λ' are bounded by $M \geq 1$ in absolute value and $|\lambda(x) - \lambda'(x)| \leq \delta$ for all $x \in \Lambda$. Then

$$\begin{aligned} |Z_\Lambda(\lambda) - Z_\Lambda(\lambda')| &\leq \sum_{k \geq 1} \frac{1}{k!} \int_{\Lambda^k} e^{-U(x_1, \dots, x_k)} \cdot \left| \prod_{i=1}^k \lambda(x_i) - \prod_{i=1}^k \lambda'(x_i) \right| dx_1 \cdots dx_k \\ &\leq \sum_{k \geq 1} \frac{1}{k!} \int_{\Lambda^k} \left| \prod_{i=1}^k \lambda(x_i) - \prod_{i=1}^k \lambda'(x_i) \right| dx_1 \cdots dx_k \\ &\leq \sum_{k \geq 1} \frac{1}{k!} |\Lambda|^k M^k k \delta \\ &\leq \delta |\Lambda| M e^{|\Lambda| M}, \end{aligned}$$

and so $Z_\Lambda(\cdot)$ is uniformly continuous. □

Lemma 11. *For any region Λ , if an activity function λ on Λ is bounded and totally zero-free then for every $v \in \Lambda$, $\rho_{\lambda'}(v)$ is uniformly continuous in λ' in a neighborhood of λ where the modulus of continuity is uniformly bounded as v varies.*

Proof. Since λ is totally zero-free, by the uniform continuity of Z , λ' is totally zero-free and $\rho_{\lambda'}(v)$ is well defined for λ' in a neighborhood of λ in the sup norm. Since $Z_\Lambda(\lambda) \neq 0$, by Lemma 9 there is some $\varepsilon > 0$ so that $|Z_\Lambda(\lambda')|$ is uniformly bounded away from 0 for λ' so that $\|\lambda - \lambda'\|_\infty < \varepsilon$. Another application of Lemma 9 shows that both $\frac{1}{Z_\Lambda(\lambda')}$ and $Z_\Lambda(\lambda' e^{-\phi(v-\cdot)})$ are uniformly continuous for $\|\lambda - \lambda'\|_\infty < \varepsilon$ and so $\rho_{\lambda'}(v) = \lambda'(v) \frac{Z_\Lambda(\lambda' e^{-\phi(v-\cdot)})}{Z_\Lambda(\lambda')}$ is uniformly continuous in this neighborhood. □

3. COMPLEX CONTRACTION

For $\lambda \in \mathbb{C}$, and $\rho : \mathbb{R}^d \rightarrow \mathbb{C}$ bounded and integrable, define

$$F(\lambda, \rho) = \lambda \exp \left(- \int_{\mathbb{R}^d} \rho(w) (1 - e^{-\phi(w)}) dw \right).$$

This function arises on the right hand side of the identity in (6). We will show that it is contractive in an appropriate sense on a complex domain. This contractive property is similar to that shown for a different function in the case of discrete two-spins systems in [29, 35].

For a positive reals s, ε we let $\mathcal{N}(s, \varepsilon) = \{y \in \mathbb{C} : \exists x \in [0, s], |y - x| < \varepsilon\}$ and $\overline{\mathcal{N}}(s, \varepsilon) = \{y \in \mathbb{C} : \exists x \in [0, s], |y - x| \leq \varepsilon\}$.

Lemma 12. *For every $\lambda_0 \in (0, e/C_\phi)$ there exists $\varepsilon > 0$ and complex neighborhoods $U_1 \subset U_2$ so that $[0, e/C_\phi] \subset U_1$ with $\overline{U_1} \subset U_2$ so that the following holds. If $\lambda \in \mathcal{N}(\lambda_0, \varepsilon)$ and $\rho(x) \in \overline{U_2}$ for all $x \in \mathbb{R}^d$, then $F(\lambda, \rho) \in U_1$.*

We will prove Lemma 12 in a sequence of steps. We will apply a change of coordinates in order to make F a contraction mapping, as was done in the discrete setting in, e.g. [32, 23, 29, 35]. With this in mind, set $\psi(x) := \log(1 + C_\phi x)$. For a function $\mathbf{z} : \mathbb{R}^d \rightarrow [0, \log(1 + e)]$, define $g_\lambda(\mathbf{z}) := \psi(F(\lambda, \psi^{-1}(\mathbf{z})))$. First, we will consider only constant functions $\mathbf{z} \equiv z$; our first main step is to show that $|g'_\lambda(z)| \leq 1$ for all z and all $\lambda \in [0, e/C_\phi]$. Thus, by definition

$$g_\lambda(z) = \log(1 + C_\phi \lambda e \cdot e^{-e^z})$$

and so

$$g'_\lambda(z) = -\frac{C_\phi \lambda e \cdot e^z \cdot e^{-e^z}}{1 + C_\phi \lambda e \cdot e^{-e^z}}.$$

Lemma 13. *For all $z \geq 0$ we have $|g'_{e/C_\phi}(z)| \leq 1$.*

Proof. Set $r = e^z \in [1, \infty)$ and note that then we may write

$$h(r) := |g'_{e/C_\phi}(r)| = \frac{e^2 r e^{-r}}{1 + e^2 e^{-r}}.$$

Compute

$$h'(r) = \frac{e^2 e^{-r}(e^{2-r} - r + 1)}{(1 + e^{2-r})^2}$$

which has a unique zero at $r = 2$, showing that $h(r) \leq h(2) = 1$. Noting $h(r) \geq 0$ completes the lemma. \square

Now we can show that for $\lambda < e/C_\phi$ we have that $|g'_\lambda|$ is *strictly* less than 1, implying that g_λ is a contraction.

Lemma 14. *For each $\lambda_0 \in [0, e/C_\phi)$ there exists $\delta > 0$ so that for all $\lambda \in [0, \lambda_0]$, we have*

$$\max_{z \geq 0} |g'_\lambda(z)| \leq 1 - \delta.$$

Proof. Set $\varepsilon = e/C_\phi - \lambda_0 > 0$. Set $\beta = C_\phi e \lambda$ so that $\beta \in [0, (1 - \varepsilon)e^2]$. Assume $\beta > 0$, as otherwise $g'_\lambda = 0$. Write

$$\begin{aligned} \max_{z \geq 0} \left| \frac{g'_\lambda(z)}{g'_{e/C_\phi}(z)} \right| &= \max_{z \geq 0} \frac{e^{-2} + e^{-e^z}}{\beta^{-1} + e^{-e^z}} \\ &= \max_{z \geq 0} \left(1 - \frac{\beta^{-1} - e^{-2}}{\beta^{-1} + e^{-e^z}} \right) \\ &= 1 - \frac{\beta^{-1} - e^{-2}}{\beta^{-1} + e}. \end{aligned}$$

Choosing $\delta := \min_{\beta \in [0, (1-\varepsilon)e^2]} \frac{1 - \beta e^{-2}}{1 + \beta e}$ completes the proof. \square

We now extend the definition of $g_\lambda(\mathbf{z})$ to take complex values of λ and complex-valued functions \mathbf{z} . Our next lemma is for the special case of constant, real \mathbf{z} .

Lemma 15. *Fix $\lambda_0 \in [0, e/C_\phi)$. Then there exists an $M > 0$ so that for all $\varepsilon_1, \varepsilon_2 > 0$ sufficiently small we have*

$$\max_{\lambda \in \mathcal{N}(\lambda_0, \varepsilon_1), z \in [-\varepsilon_2, \log(1+e)]} \left| \frac{d}{d\lambda} g_\lambda(z) \right| \leq M.$$

Proof. This follows from computing

$$\frac{d}{d\lambda}g_\lambda(z) = \frac{C_\phi e \cdot e^{-e^z}}{1 + C_\phi \lambda e \cdot e^{-e^z}}.$$

□

We now prove a version of Lemma 12 for the case of $\mathbf{z} \equiv z$:

Lemma 16. *Fix $\lambda_0 \in [0, e/C_\phi]$. There exists $\varepsilon_1, \varepsilon_2 > 0$ and $\varepsilon_3 \in (0, \varepsilon_2)$ so that for any $\lambda \in \mathcal{N}(\lambda_0, \varepsilon_1)$ and $z \in \overline{\mathcal{N}}(\log(1+e), \varepsilon_2)$ we have $g_\lambda(z) \in \mathcal{N}(\log(1+e), \varepsilon_3)$.*

Proof. First note from Lemma 14, there is a $\delta > 0$ so that $|g'_\lambda(z)| \leq 1 - \delta$ for all $\lambda \in [0, \lambda_0]$ and $z \in [0, \log(1+e)]$. Since g'_λ is continuous in z and λ , we may take $\varepsilon_1, \varepsilon_2$ sufficiently small so that

$$(7) \quad |g'_\lambda(z)| \leq 1 - \delta/2, \quad \text{for all } (z, \lambda) \in \overline{\mathcal{N}}(\log(1+e), \varepsilon_2) \times \mathcal{N}(\lambda_0, \varepsilon_1).$$

To see this, note that for each $\lambda \in [0, \lambda_0]$ and $z \in [0, \log(1+e)]$ we may find a neighborhood of (λ, z) for which the above holds, thereby giving an open cover of $[0, \lambda_0] \times [0, \log(1+e)]$; by compactness, we may reduce to a finite cover, thereby giving $\varepsilon_1, \varepsilon_2$.

By Lemma 15, we may take ε_1 small enough so that

$$(8) \quad \max_{\lambda \in \mathcal{N}(\lambda_0, \varepsilon_1), z \in [-\varepsilon_2, \log(1+e)]} \left| \frac{d}{d\lambda}g(z) \right| \leq \frac{\delta\varepsilon_2}{4\varepsilon_1}.$$

For $\lambda \in \mathcal{N}(\lambda_0, \varepsilon_1)$ and $z \in \overline{\mathcal{N}}(\log(1+e), \varepsilon_2)$, find $z' \in [0, \log(1+e)]$ with $|z - z'| \leq \varepsilon_2$ and $\lambda' \in [0, \lambda_0]$ with $|\lambda - \lambda'| \leq \varepsilon_1$. Using the bounds (7) and (8), bound

$$\begin{aligned} |g_\lambda(z) - g_{\lambda'}(z')| &\leq |g_\lambda(z) - g_\lambda(z')| + |g_\lambda(z') - g_{\lambda'}(z')| \\ &< (1 - \delta/2)\varepsilon_2 + \varepsilon_2\delta/4 \\ &= \varepsilon_2(1 - \delta/4) \\ &=: \varepsilon_3. \end{aligned}$$

Since $g_\lambda([0, \log(1+e)]) \subset [0, \log(1+e)]$ we have $g_\lambda(z) \in [0, \log(1+e)]$, thereby completing the proof. □

To generalize Lemma 16 to non-constant \mathbf{z} , we will use a convexity argument. We need the following fact that appears in [29, Proof of Lemma 4.1].

Fact 17. *For $r > 0$ and ε sufficiently small, the image of $\overline{\mathcal{N}}(r, \varepsilon)$ under the map $z \mapsto e^z$ is convex.*

This allows us to generalize Lemma 16 almost immediately.

Lemma 18. *Fix $\lambda_0 \in [0, e/C_\phi]$. There exists $\varepsilon_1, \varepsilon_2 > 0$ and $\varepsilon_3 \in (0, \varepsilon_2)$ so that for any $\lambda \in \mathcal{N}(\lambda_0, \varepsilon_1)$ and any \mathbf{z} so that $\mathbf{z}(x) \in \overline{\mathcal{N}}(\log(1+e), \varepsilon_2)$ for all x , we have $g_\lambda(\mathbf{z}) \in \mathcal{N}(\log(1+e), \varepsilon_3)$.*

Proof. We will show that there is some $z \in \overline{\mathcal{N}}(\log(1+e), \varepsilon_2)$ so that $g_\lambda(\mathbf{z}) = g_\lambda(z)$. This would follow if we find such a z so that

$$C_\phi e^z = \int_{\mathbb{R}^d} e^{\mathbf{z}(w)} (1 - e^{-\phi(w)}) dw.$$

Set $\alpha_w := (1 - e^{-\phi(w)})/C_\phi$, and note that $\alpha_w \geq 0$ with $\int \alpha_w = 1$; in particular, $\alpha_w dw$ is a probability measure. Fact 17 implies there is some such z so that $e^z = \int e^{z(w)} \alpha_w dw$. Then we can apply Lemma 16 to complete the proof. \square

Proof of Lemma 12. Find $\varepsilon_1, \varepsilon_2$ and ε_3 according to Lemma 18. Then we claim we may take $\varepsilon = \varepsilon_1$ and $U_2 = \psi^{-1}(\mathcal{N}(\log(1+e), \varepsilon_2)), U_1 = \psi^{-1}(\mathcal{N}(\log(1+e), \varepsilon_3))$. The only thing to check is that U_j are open and bounded, and the boundaries of $\mathcal{N}(\log(1+e), \varepsilon_j)$ are mapped to the boundaries of their inverse image under ψ ; since ψ^{-1} is conformal these properties hold. \square

4. ZERO FREENESS

In this section we use the tools from the previous sections to prove Theorem 5. It is contained in the following stronger theorem.

Theorem 19. *For every $\lambda_0 \in (0, e/C_\phi)$ there exists $\varepsilon > 0$ and $C > 0$ so that the following holds for every region $\Lambda \subset \mathbb{R}^d$. Let λ be an activity function on Λ and suppose $\lambda(x) \in \mathcal{N}(\lambda_0, \varepsilon)$ for all $x \in \Lambda$. Then*

$$|\log Z_\Lambda(\lambda)| \leq C|\Lambda|.$$

In particular, $Z_\Lambda(\lambda) \neq 0$. Furthermore, for every $v \in \Lambda$, $|\rho_\lambda(v)| \leq C$.

Proof. Fix Λ . Fix λ_0 and let $\varepsilon > 0$ and U_1, U_2 be as guaranteed in Lemma 12. Let $C = \max\{|z| : z \in \overline{U_2}\}$. Fix some λ so that $\lambda(x) \in \mathcal{N}(\lambda_0, \varepsilon)$ for all x . We will show that for all $\lambda' \in \mathcal{A}_\lambda$,

$$(9) \quad \rho_{\lambda'}(v) \in \overline{U_2} \text{ for all } v \in \Lambda.$$

Towards this end let $\mathcal{A}_* \subseteq \mathcal{A}_\lambda$ be the subset of activity functions λ' for which (9) holds for all $\lambda'' \in \mathcal{A}_{\lambda'}$. In particular, if $\lambda' \in \mathcal{A}_*$, then λ' is totally zero-free.

First we observe that if $\lambda' \in \mathcal{A}_*$, then since $\hat{\lambda}'_x \in \mathcal{A}_{\lambda'}$ (where $\hat{\lambda}'_x$ is as in Lemma 7), we have $|\rho_{\hat{\lambda}'_x}(x)| \leq C$. Applying Lemma 7 then gives that

$$(10) \quad |\log Z(\lambda')| \leq \int_\Lambda |\rho_{\hat{\lambda}'_x}(x)| dx \leq C|\Lambda|.$$

Our goal is now to show that $\mathcal{A}_* = \mathcal{A}_\lambda$, or equivalently that $\lambda \in \mathcal{A}_*$. The identically 0 activity function is in \mathcal{A}_* since $\rho_0(v) = 0$. Now suppose for the sake of contradiction that $\mathcal{A}_* \neq \mathcal{A}_\lambda$. Let λ_1 be an arbitrary activity function in $\mathcal{A}_\lambda \setminus \mathcal{A}_*$ and for $t \in [0, 1]$ let $\lambda_t = t\lambda_1$. We have $0 = \lambda_0 \in \mathcal{A}_*$, $\lambda_1 \notin \mathcal{A}_*$, and $\lambda_t \in \mathcal{A}_* \Rightarrow \lambda_{t'} \in \mathcal{A}_*$ if $t' < t$. We can then define $t^* = \sup\{t : \lambda_t \in \mathcal{A}_*\}$ and set $\lambda_* = \lambda_{t^*}$. By the uniform continuity of $Z(\cdot)$ and $\rho(x)$ around $\lambda \equiv 0$ (Lemmas 9 and 11), we know that $t^* > 0$.

Now for any $t < t^*$, $\lambda_t \in \mathcal{A}_*$ and so by (10), $|\log Z(\lambda_t)| \leq C|\Lambda|$. Since this is true for all $t < t^*$, by the uniform continuity of $Z(\cdot)$ we have that λ_* is totally zero-free and thus ρ is uniformly continuous in a neighborhood of λ_* . This then shows $\lambda_* \in \mathcal{A}_*$.

Moreover, by uniform continuity again, we have that λ' is totally zero-free for all λ' in a neighborhood of λ_* in the sup norm and thus $\rho_{\lambda'}(v)$ is well defined for all x and all λ' in this neighborhood.

Now consider $\rho_{\lambda_*}(x)$ for an arbitrary $x \in \Lambda$. We know that for each w , $\rho_{(\lambda_*)_{v \rightarrow w}}(w) \in \overline{U_2}$ since $\rho_{(\lambda_*)_{v \rightarrow w}} \in \mathcal{A}_{\lambda_*}$ and thus $\rho_{(\lambda_*)_{v \rightarrow w}} \in \mathcal{A}_*$. Then applying Theorem 8 and Lemma 12 we

see that $\rho_{\lambda_*}(v) \in U_1$. But then by the uniform continuity of $\rho_{\lambda_*}(x)$ in a neighborhood of λ_* and the fact that $\text{dist}(\bar{U}_1, U_2^c) > 0$, we see that $\lambda' \in \mathcal{A}_*$ for all λ' in a neighborhood of λ_* . In particular, $\lambda_{t'} \in \mathcal{A}_*$ for some $t' > t_*$, a contradiction. Thus we have that $\mathcal{A}_* = \mathcal{A}_\lambda$ as desired.

Now since $\lambda \in \mathcal{A}_*$, applying (10) again gives $|\log Z(\lambda)| \leq C|\Lambda|$ as desired. \square

5. PROOF OF THEOREM 1, THEOREM 2, AND COROLLARY 4

To prove analyticity of the pressure and uniqueness, we use Vitali's Theorem (see, for instance, [36, Theorem 6.2.8]):

Theorem 20 (Vitali's Convergence Theorem). *Let $\Omega \subset \mathbb{C}$ be a complex neighborhood and f_n a sequence of analytic functions on Ω so that $|f_n(z)| \leq M$ for some M and for all n and all $z \in \Omega$. If there is a sequence $z_m \in \Omega$ with $z_m \rightarrow z_\infty \in \Omega$ so that $\lim_{n \rightarrow \infty} f_n(z_m)$ exists for all m , then f_n converges uniformly on compact subsets of Ω to an analytic function f on Ω .*

Proof of Theorem 1. We first prove analyticity of the pressure. By Theorem 5, the finite volume pressure $p_V(\lambda) := \frac{1}{V} \log Z_{\Lambda_V}(\lambda)$ is an analytic function of λ in L_0 and uniformly bounded. For $\lambda \geq 0$ $p_V(\lambda)$ converges as $V \rightarrow \infty$ (see, e.g., [34]). Thus, by Vitali's Theorem (Theorem 20), $p_V(\lambda)$ converges uniformly to an analytic function in L_0 .

As explained in [34, 18], to prove uniqueness of the infinite volume Gibbs measure, it is enough to show that for all $k \geq 1$ and all $(v_1, \dots, v_k) \in (\mathbb{R}^d)^k$, there exist functions $\rho_{\infty, \lambda}(v_1, \dots, v_k)$ so that for all cofinal sequences of regions $\Lambda_n \rightarrow \mathbb{R}^d$,

$$(11) \quad \lim_{n \rightarrow \infty} \rho_{\Lambda_n, \lambda}(v_1, \dots, v_k) = \rho_{\infty, \lambda}(v_1, \dots, v_k).$$

This is true because when correlation functions satisfy the Ruelle bound (in the repulsive case we have $\rho_{\Lambda, \lambda}(v_1, \dots, v_k) \leq \lambda^k$), the collection of k -point correlation functions determines the distribution of a point process. For $\lambda \in \mathbb{C}$, $|\lambda| < 1/(eC_\phi)$, (11) holds for all $k \geq 1$ and all $(v_1, \dots, v_k) \in (\mathbb{R}^d)^k$ by [34, Theorem 4.2.3].

Now write

$$\begin{aligned} \rho_{\Lambda, \lambda}(v_1, \dots, v_k) &= \lambda^k e^{-U(v_1, \dots, v_k)} \frac{Z_\Lambda(\lambda e^{-\sum_{i=1}^k \phi(v_i - \cdot)})}{Z_\Lambda(\lambda)} \\ &= \lambda^k e^{-U(v_1, \dots, v_k)} \prod_{j=1}^k \frac{Z_\lambda(\lambda_j)}{Z_\lambda(\lambda_{j-1})} \\ &= \prod_{j=1}^k \rho_{\Lambda, \lambda_j}(v_j) \end{aligned}$$

where $\lambda_j = \lambda e^{-\sum_{i=1}^j \phi(v_i - \cdot)}$ (in particular $\lambda_0 \equiv \lambda$). Then by (4) and Theorem 19 each density $\rho_{\Lambda, \lambda_j}(v_j)$ is an analytic function of λ and uniformly bounded and so $\rho_{\Lambda, \lambda}(v_1, \dots, v_k)$ is analytic and uniformly bounded in Λ . By Vitali's Theorem, we then have that for every (v_1, \dots, v_k) , $\rho_{\Lambda_n, \lambda}(v_1, \dots, v_k)$ converges to an analytic function of λ in $\mathcal{N}(e/C_\phi, \varepsilon)$. By the identity theorem for analytic functions, the limit must be unique since the limit is unique for $|\lambda|$ sufficiently small (see [34]). \square

To prove Theorem 2 we use a lower bound on the density as a function of λ for λ positive, closely related to an inequality of Lieb [24, Eq. (1.19)]. Fix the potential ϕ , and for a region Λ , let $\rho_\Lambda(\lambda) = \lambda \frac{d}{d\lambda} \frac{1}{|\Lambda|} \log Z_\Lambda(\lambda)$ be the finite volume density.

Lemma 21. *For any repulsive, tempered potential ϕ , any region Λ , and any $\lambda \geq 0$, $\rho_\Lambda(\lambda) \geq \lambda/(1 + \lambda C_\phi)$.*

The proof below is a generalization of that of [16, Lemma 18] for the case of hard spheres.

Proof. We denote by \mathbf{X} the random point set in Λ drawn from the Gibbs measure $\mu_{\Lambda, \lambda}$, with density $e^{-U(\cdot)}$ against the Poisson process of intensity λ on Λ . The finite volume density is then $\frac{1}{|\Lambda|} \mathbb{E} |\mathbf{X}|$.

We will use a simple consequence of inclusion–exclusion, that for each n and every sequence of numbers x_1, \dots, x_n , with $x_j \in [0, 1]$ for all j , we have $\prod_{j=1}^n (1 - x_j) \geq 1 - \sum_{j=1}^n x_j$.

Now since $\rho_\Lambda(\lambda) = \lambda \frac{Z'(\lambda)}{|\Lambda| Z(\lambda)}$ we have

$$\begin{aligned}
\rho_\Lambda(\lambda) &= \frac{\lambda}{|\Lambda| Z_\Lambda(\lambda)} \sum_{k \geq 0} \int_{\Lambda^{k+1}} \frac{\lambda^k}{k!} \prod_{0 \leq i < j \leq k} e^{-\phi(x_i - x_j)} dx_1 \dots dx_k dx_0 \\
&= \frac{\lambda}{|\Lambda| Z_\Lambda(\lambda)} \int_{\Lambda^k} \prod_{1 \leq i < j \leq k} e^{-\phi(x_i - x_j)} \frac{\lambda^k}{k!} \left(\int_{\Lambda} \prod_{j=1}^k e^{-\phi(x_j - x_0)} dx_0 \right) dx_1 \dots dx_k \\
&= \frac{\lambda}{|\Lambda|} \mathbb{E} \left[\int_{\Lambda} \prod_{y \in \mathbf{X}} e^{-\phi(y-x)} dx \right] \\
&= \frac{\lambda}{|\Lambda|} \mathbb{E} \left[\int_{\Lambda} \prod_{y \in \mathbf{X}} (1 - (1 - e^{-\phi(y-x)})) dx \right] \\
&\geq \frac{\lambda}{|\Lambda|} \mathbb{E} \left[\int_{\Lambda} 1 - \sum_{y \in \mathbf{X}} (1 - e^{-\phi(y-x)}) dx \right] \\
&\geq \lambda \left(1 - \mathbb{E} \left[\frac{\mathbf{X}}{|\Lambda|} C_\phi \right] \right) \\
&= \lambda - \lambda C_\phi \rho_\Lambda(\lambda).
\end{aligned}$$

Rearranging gives the lemma. □

With this we can prove Theorem 2.

Proof of Theorem 2. Fix $\lambda_0 \in [0, e/C_\phi)$. Then by Theorem 19, there is an $\varepsilon > 0$ so that for all regions Λ the pressure $\frac{1}{V} \log Z_{\Lambda_V}(\lambda)$ is analytic and bounded above for $\lambda \in \mathcal{N}(\lambda_0, \varepsilon)$; by the proof of Theorem 1, the finite volume pressure converges uniformly on compact subsets of $\mathcal{N}(\lambda_0, \varepsilon)$ and so its derivative with respect to λ (multiplied by λ) converges uniformly on compact subsets to the limit $\rho_\phi(\lambda)$. In order to show that p_ϕ may be taken as an analytic function of ρ_ϕ , it is sufficient to show that λ may be taken as an analytic a function of ρ_ϕ . To show this, we will use the inverse function theorem for analytic functions (e.g., [36, Theorem 2.1.1]). Since ρ_ϕ is an analytic function of λ , it is sufficient to show that $\rho'_\phi(\lambda) \neq 0$

for $\lambda \in [0, \lambda_0]$. This follows from, e.g., inequality (5) in [12] which gives a uniform lower bound on the derivative of the finite volume density with respect to λ . This shows that for $\lambda \in [0, \lambda_0]$ we have that $\rho'_\phi(\lambda) \neq 0$ and so ρ_ϕ^{-1} is analytic in a complex neighborhood of $[0, \rho_\phi(\lambda_0)]$. By Lemma 21, this interval contains the interval $\left[0, \frac{\lambda_0}{1+\lambda_0 C_\phi}\right]$. Sending $\lambda_0 \rightarrow e/C_\phi$ proves analyticity of p_ϕ in ρ_ϕ in the interval $\left[0, \frac{e}{1+e} \frac{1}{C_\phi}\right)$. \square

To conclude we prove Corollary 4.

Proof of Corollary 4. The first statement is an immediate consequence of Theorem 2. The second statement follows by replacing the lower bound on the density from Lemma 21 with the bound, specific to the hard sphere model, from [16, Theorem 19] (which in turn comes from [20]), which says that for λ fixed, the packing density in dimension d at fugacity λ is at least $(1 + o_d(1))W(\lambda)2^{-d}$, where $W(\cdot)$ is the W-Lambert function. In particular, $W(e) = 1$, which gives Corollary 4. \square

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