GIBBS MEASURES, SPARSE GRAPHS, AND OPTIMIZATION

WILL PERKINS

Workshop on Graph limits in Bohemian Switzerland 2018[1]

This minicourse gives an introduction to statistical physics models on sparse graphs and develops some methods for applying these models to other areas of mathematics. The main theme of the course is that partition functions from statistical physics can express many important graph theoretic parameters, and by a combination of sampling and optimizing, we can bound logarithmic derivatives of these partition functions. We will apply this idea both to graph theoretic problems and to geometric problems in Euclidean space. Along the way we will present several central open problems in mathematical physics, probabilistic and extremal combinatorics, and discrete geometry.

Lecture 1: Gibbs measures, phase transitions, and sparse graphs. We begin by introducing one of the most simple but mysterious models of a gas or crystal from statistical physics, the *hard sphere model*. We give a mathematical definition of a *phase transition*. We introduce two discrete statistical physics models: the *hard-core model* (or hard-core lattice gas) of a random independent set from a graph, and the *monomer-dimer model* of a random matching from a graph. All of the above models are *Gibbs measures*, and we introduce some basic objects and properties of Gibbs measures: partitions functions, free energies, and the spatial Markov property. We survey three settings of interest for Gibbs measures on bounded degree graphs: lattices (like $\mathbb{Z}^d$) in statistical physics; random graphs (like $G(n, d/n)$ in probabilistic combinatorics and computer science); and extremal graphs (graph theory and combinatorics)

Lecture 2: Independent sets and occupancy fractions. We introduce a physical observable of the hard-core model, the *occupancy fraction*, the expected fraction of vertices that appear in a random independent set drawn from the model. We describe some of its properties, including the fact that it is the scaled derivative of the log partition function. We then present a method for proving upper and lower bounds on the occupancy fraction over various classes of graphs via linear programming relaxations.

Lecture 3: The Potts model and the monomer-dimer model. We set up linear programming relaxations to maximize or minimize observables associated to the Potts model and monomer dimer model in regular graphs. This allows us to prove results on the number of matchings and number of q-colorings in regular graphs. The resulting linear programs seem difficult to solve analytically. We also present a number of open problems.

Lecture 4: Sphere packings, kissing numbers, and spherical codes. We apply a variant of the independent set occupancy method in a continuous setting to sphere packings in Euclidean space and packings of spherical caps on a d-dimensional sphere. Consequences of these bounds include a lower bound on the entropy of sphere packings of density $\Theta(d \cdot 2^{-d})$ and a new lower bound on the kissing number in high dimensions.

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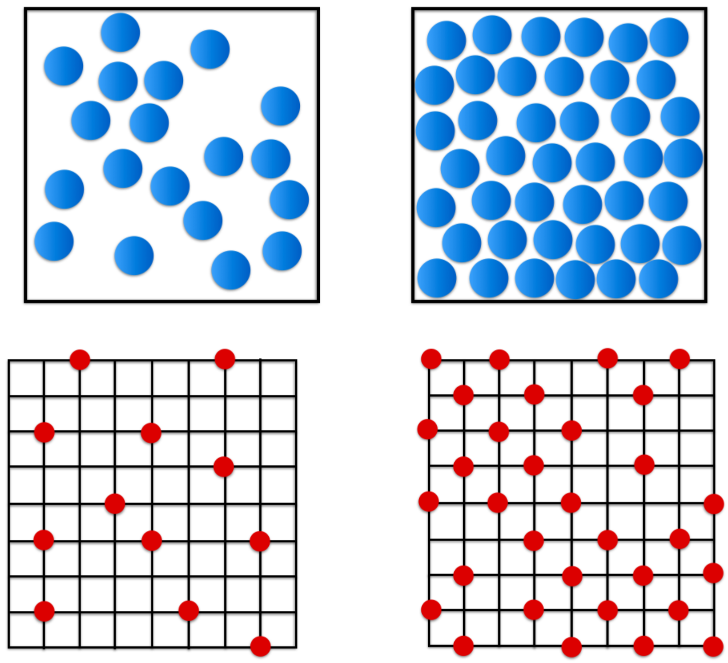

Background reading. The textbook of Mezard and Montanari [12] gives a good introduction to the principles of statistical physics and the study of Gibbs measures on random graphs. Brightwell and Winkler [1] describe connections between statistical physics and combinatorics. Yufei Zhao has a nice survey article on extremal problems for sparse graphs [14]. David Galvin has lecture notes on the entropy method [8]. Harmut Löwen gives an introduction to the hard sphere model [11]. Henry Cohn surveys sphere packing bounds [2] and describes very recent developments [3]. Some of the results we present in the course come from recent papers [4, 5, 6, 7, 9, 10, 13].

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