CS 8803: Statistical Physics in Algorithms and Combinatorics: Assignment 1

Due Wednesday, January 22. Type solutions and send by email to wperkins3@gatech.edu

(1) Let $\mathcal{G}_n$ be the set of all (labeled) graphs on n vertices. For $m \in (0, \binom{n}{2})$, determine the maximum entropy probability distribution on $\mathcal{G}_n$ with mean number of edges m. (Recall that the entropy of a probability distribution μ on a finite set Ω is $H(\mu) = -\sum_{x\in\Omega} \mu(x)\log\mu(x)$ with the convention that $0 \log 0 = 0$).

(2) Let K_d be the complete graph (clique) on d vertices.
  (a) Compute the hard-core partition function $Z_{K_d}(\lambda)$.
  (b) For $u, v \in K_d$ compute the truncated two-point correlation function.

(3) Let $G = G_1 \cup G_2$, the disjoint union of two graphs G_1, G_2. Prove that

$$Z_G(\lambda) = Z_{G_1}(\lambda) Z_{G_2}(\lambda)\,.$$

(4) Let $K_{d,d}$ be the complete d-regular bipartite graph (two sets of d vertices L, R) with all d^2 edges between L and R present and no others).
  (a) Compute $Z_{K_{d,d}}(\lambda)$.
  (b) Compute $\mathbb{E}_{K_{d,d},\lambda}|I|$, the expected size of an independent set I drawn from the hard-core model on $K_{d,d}$ at fugacity λ.

(5) Prove that the following probability distribution on independent sets of G is the hard-core model on G at fugacity λ. Pick a subset $S \subseteq V(G)$ by including each vertex independently with probability $\frac{\lambda}{1+\lambda}$ and condition on the event that S is an independent set.

(6) Consider the hard-core model on a graph G and let F be the set of vertices that are not in the independent set and have no neighbor in the independent set (they are free to be added to the independent set). Calculate $\mathbb{E}[|F|]$ in terms of derivatives of $\log Z_G(\lambda)$.

(7) Let P_n be the path on n vertices.
  (a) Write a recursion for the independence polynomial $Z_{P_n}(\lambda)$.
  (b) Solve the recursion to compute the limit $f(\lambda) = \lim_{n\to\infty} \frac{1}{n} \log Z_{P_n}(\lambda)$.
  (c) What can you deduce about phase transitions in the hard-core model on $\mathbb{Z}$ from the function $f(\lambda)$?
  (d) Prove that for the hard-core model on $\mathbb{Z}$, the truncated two-point correlation function decays exponentially fast in the distance, for any $\lambda > 0$.