CS 8803: Statistical Physics in Algorithms and Combinatorics: Assignment 2

Due Wednesday, February 5. Type solutions and send by email to wperkins3@gatech.edu

(1) Fix $d > 0, q \geq 3$. Let $p = d/n$. Let $G \sim G(n,p)$ and let $Z_q(G)$ denote the number of proper q-colorings of G.
  (a) Compute $\mathbb{E} Z_q(G)$.
  (b) Using the first-moment method (and the calculation above), find $d_1 > 0$ so that if $d > d_1$, then whp G is not q-colorable.
  (c) Improve this bound by conditioning on a high probability event (based on the number of edges of G). Write the new threshold $d_2 < d_1$ explicitly.
  (d) Prove that for $d < 1$, the following holds:

$$\frac{1}{n} \log Z_q(G) \to \log q + \frac{d}{2} \log \left(1 - \frac{1}{q}\right)$$

  in probability as $n \to \infty$. (Hint: use a fact about the random graph $G(n,p)$).
  (e) Assume for simplicity that q divides n. A proper q-coloring is balanced if the color classes are all the same size. Let $Z_{q,\mathrm{bal}}(G)$ denote the number of balanced proper q-colorings of G. For the following questions, work with the $G(n,m)$ model with $m = dn/2$.
    (i) Show that $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} Z_{q,\mathrm{bal}}(G) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} Z_q(G)$
    (ii) Let $f_2(q,d) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[Z^2_{q,\mathrm{bal}}(G)]$. Express $f(q,d)$ as an optimization problem over doubly stochastic $q \times q$ matrices.
    (iii) State the Paley–Zygmund inequality.
    (iv) Assume that for some d, q the unique optimizer of this optimization problem above is the constant $1/q^2$ matrix. What can you conclude about the probability that G has a proper q-coloring?

(2) Consider the ferromagnetic Ising model (i.e. $Z = \sum_\sigma \lambda^{\#+1 \text{ spins}} e^{\beta \cdot \# \text{ monochromatic edges}}$) with inverse temperature β and external field λ on the tree T shown below. Take $\lambda = 2$ and $e^\beta = 3$.

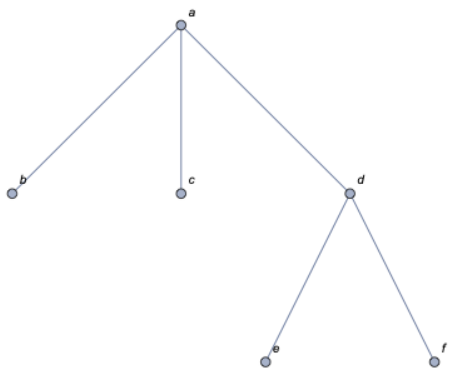

FIGURE 1. The tree T

  (a) Compute all of the BP messages sent from each vertex to each of its neighbors.
  (b) Using these messages, compute the marginals of the Ising model for each vertex.
  (c) Using the messages, write down the Bethe free energy for $\log Z_T$.

(3) Define the function $g_c(x) = c \exp(-x)$.
  (a) Show that for $c > 0$ fixed, g_c has a unique fixed point on the interval $[0, c]$, call this x_c. Give a formula for x_c.

(b) Show that there is some $c^* > 0$ so that if $c < c^*$, then starting from any $x_0 \in [0, c]$, the iteration $x_{j+1} = g_c(x_j)$ converges to x_c. Give a formula for the best possible such c^*.
(c) Show that for $c > c^*$, the iteration may not converge.
(d) Use the above to prove the following facts about phase transitions in the hard core model on the infinite Δ-regular tree $\mathbf{T}_\Delta$.
  (i) If $c < c^*$, then for sufficiently large Δ, and $\lambda = c/\Delta$, there is a unique λ-hard-core measure on $\mathbf{T}_\Delta$.
  (ii) If $c > c^*$, then for sufficiently large Δ, and $\lambda = c/\Delta$, there are multiple distinct λ-hard-core measures on $\mathbf{T}_\Delta$.
(e) (Bonus question) Can you carry out the same kind of analysis for $\mathbf{T}_\Delta^{(3)}$, the infinite Δ-regular, 3-uniform, linear hypergraph? (A hypergraph independent set is a set of vertices that induce no hyperedges. A linear hypergraph is one in which two edges overlap in at most one vertex.)