Math 3215: Lecture 16

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1 The Central Limit Theorem

Let $X_1, \dots X_n$ be i.i.d. random variables with mean μ and variance σ^2. Let $Y = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$. Then

$$\lim_{n\to\infty} \Pr[s \leq Y \leq t] = \Phi(t) - \Phi(s)$$

Some values of $\Phi(t)$:

- $\Phi(1) - \Phi(0) = .34$
- $\Phi(2) - \Phi(0) = .477$
- $\Phi(3) - \Phi(0) = .4987$

How to use the CLT?

- Say the height of a random American has mean 5.5 ft and variance .25. If we take the average height of 100 people, estimate the probability that this average will be greater than 6 ft.
- You can express these probabilities in terms of Z, a standard normal rv. eg. $\Pr[Z \leq 1.5]$ or $\Pr[-.5 \leq Z \leq 1.5]$. The trick is applying a transformation so that the r.v. you're interested in now has mean 0 and standard deviation 1.
- How? 1) Subtract the mean. 2) Divide by standard deviation.
- Do plenty examples of this. This is a basic step in many statistics problems.

2 Why is the CLT true?

Let X and Y be independent, random variables that take the values $-1, 1$ with probability 1/2. Let $\sigma^2 = var(X)$.

- What is σ^2?
- What is $\mathbb{E}[X + Y]$?
- What is $var(X + Y)$?
- Find the probability mass function for $Z = (X + Y)/sd(X + Y)$.
- What is $\mathbb{E}Z$ and $var(Z)$?

Now let X and Y be independent, uniform $[-1, 1]$ random variables. Let $\sigma^2 = var(X)$.

- What is σ^2?
- What is $\mathbb{E}[X+Y]$?
- What is $var(X+Y)$?
- Find either the density function or CDF for $Z = (X+Y)/sd(X+Y)$

Now let X and Y be independent, $N(0,1)$ rv's. Let $\sigma^2 = var(X)$.

- What is σ^2?
- What is $\mathbb{E}[X+Y]$?
- What is $var(X+Y)$?
- Find either the density function or CDF for $Z = (X+Y)/sd(X+Y)$

3 A Proof of the CLT

def. The *characteristic function* of a random variable X is

$$\phi_X(t) := \mathbb{E}e^{itX}$$

where $i = \sqrt{-1}$. For a continuous rv, $\phi_X(t) = \int_{-\infty}^{\infty} e^{itX} f(x)\,dx$.

Examples:

- Let $X = +1, -1$ wp $1/2$. What is $\phi_X(t)$?
- Let X be uniform on $[0,1]$. What is $\phi_X(t)$?
- Let $X \sim N(0,1)$. What is $\phi_X(t)$? answ: $e^{-t^2/2}$.

Facts about characteristic functions:

- If $\phi_{X_n}(t) \to \phi_X(t)$ for every t, then $X_n \to X$, i.e. $\Pr[X_n \leq s] \to \Pr[X \leq s]$ for all s at which $F_X(s)$ is continuous. and vice-versa.
- If X and Y are independent rv's with characteristic functions $\phi_X(t)$ and $\phi_Y(t)$ respectively, then what is $\phi_{X+Y}(t)$?
- What is $\phi_{cX}(t)$?

Steps in the proof of the CLT:

1. Know the power series expansion $e^x = 1 + x + x^2/2 + x^3/3! + \ldots$
2. Know the limit $(1 + y/n)^n \to e^y$
3. Let $X_1, \ldots X_n$ be iid w characteristic function $\phi_X(t)$, mean 0 and variance σ^2. Let $Y_n = \frac{X_1 + \cdots + X_n}{\sigma\sqrt{n}}$.
4. What is $\phi_{Y_n}(t)$?
5. Fix t, take limit: $\lim_{n\to\infty} \phi_{Y_n}(t)$. What do you get?