Math 3215: Lecture 24

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1 Random Graphs

1.1 What's a Graph

A *graph* is a network: a collection of vertices (nodes) and edges (connections between nodes). We can draw a graph with dots and lines, but the particular way we draw a graph doesn't matter - the graph itself is completely determined by which vertices are connected to which other vertices.

Examples of things we can model with graphs:

- Road networks: each city is a vertex and we add an edge between cities if there is a road connecting them. We can create a *weighted graph* by giving each edge a weight corresponding to the road's distance.
- Facebook. A social network can be represented by a graph where every person is a vertex and there is an edge if two people are friends.
- A computer network. The internet itself can be represented by a graph where every device (computer, router, server) is a vertex and there is an edge if two devices are directly connected.
- A sports tournament. Each team is a vertex and there is an edge between two teams if they have played a game. We can make a *directed graph* by directing each edge from the winning team to the losing team.
- Chemistry. Each atom is a vertex and two atoms have an edge between them if they are joined by a molecular bond.
- Biology. An evolutionary tree is a special kind of a graph: a *tree* is a graph without cycles. Each species is a vertex in an evolutionary tree and an edge represents evolutionary descent.
- many, many other examples. The power of graphs is that they abstract away unnecessary information and leave only binary relationship information.

1.2 What kind of questions do we ask about graphs?

- How many vertices, how many edges?

- Is the graph connected (every vertex joined by a path of some length to every other vertex)?
- What is the largest clique? (set of vertices so that every vertex in the set has an edge to every other. a triangle is a clique of size 3)
- What is the largest connected component?
- What is the degree distribution? (the degree of a vertex is its number of neighbors)
- What is the smallest cut in the graph? (a cut is a way of separating the graph into two sets of vertices - the smallest cut is the cut that breaks the fewest number of edges)
- Algorithms: on very large graphs (with n vertices, and n very very large), how do we 1) represent a graph in a computer 2) answer some of the questions above efficiently ?

Example:

- How can you find the shortest path between vertices u and v in a large graph?

1.3 Random Graphs

What do graphs have to do with probability? Plenty. Here's one connection. In 1959, Paul Erdős and Alfred Rényi defined a *random graph*, $G(n, p)$. This a a graph with n vertices where each of the $\binom{n}{2}$ possible edges in the graph is present independently with probability p. p can either be a fixed number, say $1/2$, or depend on n: $p = \frac{1}{n}$ for example.

Probability questions:

- What is the expected degree of a given vertex v?
- What is the variance of the degree of a vertex?
- With $p = c/n$ what is the approximate distribution of the degree of a given vertex?
- What is the expected number of common neighbors of vertices u and v?
- With $p = c/n$ for c fixed, what is the expected fraction of vertices which are isolated (not joined to any other vertices)?
- What is the expectation and variance of the number of triangles in $G(n, p)$?
- What is the MLE estimator of p if you are given a graph G with n vertices and told it comes from $G(n, p)$ but p is an unknown parameter?