Math 3215: Lecture 3

Will Perkins

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Coins, Cards, and Counting

This lecture will all be about computing probabilities in models where all outcomes are equally likely. This involves counting outcomes in the sample space and in events of interest. The science of counting is called *combinatorics*.

1 Lists and License Plates

The most basic kind of counting is done 'with replacement' and a prime example is counting telephone numbers or counting license plates. Typical questions:

- How many 7 digit phone numbers are there? How many 7 digit phone numbers that do not begin with '1'?
- Say a license plate has 3 letters followed by 3 numbers. How many possible license plates are there?

The key properties of this type of counting problem:

- The order matters: 777-5521 is a different phone number than 521-7775.
- You can repeat letters / digits as you wish. 555-5555 is a perfectly valid phone number.

How to count? Draw boxes, multiply numbers.

2 Ranking items

The next kind of counting is counting the number of ways to arrange a collection of (distinguishable) items.

- How many different ways are there to rank 5 movies, best to worst?
- How many different seating arrangements are there for a class of 30 students?
- How many ways can we write the triangle (ABC)?

Key features:

- Again order matters. In fact that's all that matters in this case.
- No repetition allowed.

Again we can draw boxes and multiply the numbers, but with each box, the number of choices decreases by 1. This is how we get the formula $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$.

What if we just want to rank our top 3 choices out of 10 movies? The same principle holds, but the numbers in the boxes end before 1.

- How many possible pairs of president and vice-president are there in a class of 40?

3 Over-counting

The more advanced counting deals with over-counting. Let's say we want to line up 6 vegetables in order, but 2 of them are indistingusihable carrots. How different ways are there? If all 6 vegetables were distinguishable, this would simple be a ranking problem and the answer would be $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. But since the carrots are indistinguishable we've over-counted. How many times have we counted each valid outcome? Twice. Once when carrot 1 came first, and once when carrot 2 came first. So the correct answer is $\frac{6!}{2}$.

- What would we divide by if there were 4 indistinguishable carrots?

Features:

- Saying a group of items is indistinguishable is saying that within that group order does not matter.
- An extreme version of over-counting is saying that order doesn't matter: instead of ranking we are picking or choosing.

More over-counting problems:

- How many ways are there to pick 3 vice-presidents out of a class of 20?
- How many ways are there to punch 6 numbers on a lottery card with 20 chocies?
- How many ways can we flip 6 coins and get 3 heads?

4 Tips

- You may have memorized the formulas ${}_nC_k$ or ${}_nP_k$ at some point. These are fine, they are formulas fo the most common counting problems above, but I highly recommend using the method of boxes and accounting for your over-counting instead.
- It's important to be very clear in a probability problem what your set of outcomes is. In combinatorics-based problems, the right choice is the choice that makes each outcome equally likely. (Otherwise you can't solve it by counting).

5 Questions

1. come up with your own examples of each of the types of counting illustrated above.
2. What's the probability the first three cards drawn from a deck are all red?
3. What's the probability that 2 of the first three card are Aces?
4. Binomial theorem. Write $(x + y)^n$ as a sum, $\sum_{k=0}^{n}$
5. How many 7-digit phone numbers are there that don't start with 9 or 1 and that don't repeat digits?
6. What's the probability of drawing 4-of-a-kind in a 5-card poker hand?
7. How many 5-card hands of a 52-card deck have 5 distinct face values?
8. If we have r red coins and b blue coins, how many differnt ways are there to line them up? (same color coins are indistinguishable)
9. What's the probability of throwing exactly 5 heads in 12 flips of a fair coin?
10. How many different 'straight' hands are there in poker? (eg. (8-9-10-J-Q)).
11. Write your own counting problems.