Math 3215: Lecture 9

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February 7, 2012

1 Indepedent Random Variables

Definition: two random variables X and Y are *independent* if the events $X \leq t$ and $Y \leq s$ are independent for all real numbers s and t.

equivalently:

X and Y are independent if $\Pr[X \leq t \text{ and } Y \leq s] = F_X(t) \cdot F_Y(s)$ for all s, t.

- Prove that the indicator rv's for two independent events A and B are independent.
- Prove that the number of heads in 4 flips of a fair coin is not independent of the indicator rv that the 4th flip is a head.

2 Expectation

The *expectation* of a random variable is a kind of weighted average, weighted according to the probability mass function. We define the expectation of a r.v. Y:

$$\mathbb{E}Y = \sum_{\text{outcomes } x \in S} p(x)Y(x)$$

which is the same as

$$\mathbb{E}Y = \sum_{t : \Pr[Y=t] \neq 0} P_Y(t)t$$

- show that the two definitions are the same
- *note*: the expectation is *not* the most likely value Y takes, it is an average value.

Examples:

- We flip a fair coin. Let $X = 10$ if we get heads, -4 if we get tails. What is $\mathbb{E}X$?
- Let Y be the number on a single roll of a die. What is $\mathbb{E}Y$?
- Let X be the sum of the numbers on two dice. What is $\mathbb{E}Y$?
- Let $Z = 1$ if we roll a 6 on a fair die, 0 otherwise. $\mathbb{E}Z = ?$
- A hot dog man makes \$100 if it is sunny, \$50 if it is not. $\Pr[\textit{sunny}] = .8$. What is his expected earnings?

3 Expectation and Fair Games

Expectation decribes our intuitive idea of a fair game. Let's say we play a game: you roll two dice and if two two numbers are the same you win \$20 from me, and if they are different you lose \$$x$ dollars.

- What should x be for the game to be *fair*?
- Can you come up with a rigourous definition of a fair betting game? Are the games in a casino fair?

4 Expectation is Linear

A linear function is a function f so that $f(ax + y) = af(x) + f(y)$.

Expectation is also linear:

$$\mathbb{E}[aX + Y] = a\mathbb{E}X + \mathbb{E}Y$$

Prove it!

- What is the expectation of I_A, the indicator random variable for the event A?
- Let Y be the number of heads in n flips of a p-biased coin. We howed before that we can break up $Y = X_1 + \cdots + X_n$ where X_i is the indicator r.v. that flip i is a head. Use linearity to calculate $\mathbb{E}Y$.
- If X and Y are dependent, do we still have $\mathbb{E}[X + Y] = \mathbb{E}X + \mathbb{E}Y$? If not, give a counterexample.
- If 53% of people prefer coke to pepsi, and we randomly sample 120 people, what is the expected number of people in our sample who prefer coke?
- If you repeat a fair game 100 times, is that also a fair game?

5 Variance

The variance of a random variable is defined as:

$$var(X) = \mathbb{E}[(X - \mathbb{E}X)^2]$$

this is the same as:

$$var(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2$$

prove that.

Calculate some variances:

- Let $Y = 7$ with probability 1. $var(Y) =?$
- Let X_i be the indicator r.v. of a getting heads with a fair coin. $var(X_i) =?$
- X_i is the indicator of a p-biased coin. $var(X_i) =?$
- If X and Y are independent, prove that $\mathbb{E}[XY] = \mathbb{E}X\mathbb{E}Y$. (*note*: this does not go the other way)

Final question to test everything you learned today:

Let Y be the number of heads in 10 flips of a fair coin.

- What is $\mathbb{E}Y$?
- What is $var(Y)$?