Math 3215: Midterm 2 Sample Questions

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1 Continuous Random Variables

1. Let X be a random variable with density function $c_1 e^{-c_2 x}$ for $x \geq 0$. Assume $\mathbb{E}X = 2$.
   - what are c_1 and c_2?
   - Write the CDF of X.
   - Calculate $\Pr[3 \leq X \leq 5]$
2. Let X and Y be independent normal random variables with means 1 and 3 and variances 4 and 9 respectively.
   - Find the mean and variance of $Q = 2X + Y$
   - Prove that $Q = 2X + Y$ has a normal distribution.

2 Covariance

1. Let X_1, X_2 be independent, each taking the values $-1, 1$ with probability $1/2$. Let $Y_1 = 2X_1 - X_2$ and $Y_2 = 2X_2 - X_1$.
   - What is $cov(X_1, X_2)$?
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2. Express the variance of $X_1 + X_2 + X_3$ in terms of only variances and covariances, not expectations.
3. What is the covariance of the number of Red cards and the number of Black cards in a 5 card hand of poker?

   answer: Let R and B be the respetive random variables. $R = R_1 + \cdots + R_5$ where R_i is the indicator that the ith card is red. $\mathbb{E}R = \mathbb{E}B = 5/2$. We need to calculate $\mathbb{E}[RB]$.

   $$\mathbb{E}[RB] = \mathbb{E}[\sum i, j R_i B_j]$$

   $$= 5\mathbb{E}[R_1 B_1] + 20\mathbb{E}[R_1 B_2]$$

   by symmetry. $\mathbb{E}[R_1 B_1 = 0]$ and $\mathbb{E}[R_1 B_2] = \frac{1}{2} \cdot \frac{26}{51} = \frac{13}{51}$. So $\mathbb{E}[RB] = \frac{260}{51}$ and

   $$cov(R, B) = \frac{260}{51} - \frac{25}{4} = -\frac{235}{204}$$

3 Poisson Distributions

1. Let X be Poisson with mean λ.
   - Find the value of j that maximizes $\Pr[X = j]$.
   - For what value of λ does $\Pr[X = 0] = \Pr[X = 1]$

answer:

(a) $\Pr[X = k]$ increases as k goes up from 0 until it hits its maximum, then it declines. (the function is unimodal). So we need to find the first k so that $\Pr[X = k] > \Pr[X = k+1]$.

$$\begin{aligned}\Pr[X = k] - \Pr[X = k+1] &= \frac{e^{-\lambda}\lambda^k}{k!} - \frac{e^{-\lambda}\lambda^{k+1}}{(k+1)!} \\ &= \frac{e^{-\lambda}\lambda^k}{k!}\left[1 - \frac{\lambda}{k+1}\right]\end{aligned}$$

So if $k + 1 < \lambda$, $\Pr[X = k+1] > \Pr[X = k]$. So the maximum occurs at $\lceil \lambda \rceil$.

(b) $\lambda = 1$

4 Central Limit Theorem

1. Estimate as best you can (calculate a decimal) the probability that in 3600 flips of a fair coin you get less than 1860 heads.

5 Maximum Likelihood

1. Find the maximum likelihood estimator for the mean of a sequence of exponential random variables.

6 Parameter Estimation

Answers can be written in terms of probabilities of Z or T_n.

1. You're measuring the length of randomly captured butterflies, and you've caught 4, with lengths 4.8, 6.2, 5.4, 5.6 cm. What can you say about μ, the true mean length of a random butterly?

2. You take two separate random polls of voters in Idaho. In the first, 43 out of 100 people say they have a good impression of the Democratic candidate. In the second, 48 out of 100 say they have a good impression of the Republican candidate. You're asked: "Do voters have a better impression of the Democrat or the Republican?" What can you say, and with what confidence?