Math 4221: Test #2

October 3, 2012

1 Conditional Distributions

1.1

Let X and Y be independent Poisson random variables with mean 1. Conditioned on $X+Y=2$, are X and Y independent? Prove it.

They are not independent. Here's a quick way to prove it. Conditioned on $X+Y=2$, there is some positive probability that $X=0$ and some positive probability that $Y=0$. But there is 0 probability that $\{X=0 \wedge Y=0\}$. So the random variables are not independent.

1.2

Let X be an exponential random variable with mean 1 (density function e^{-x}).

What is $\Pr[X \geq 3 | X \geq 1]$?

$$\begin{aligned}
\Pr[X \geq 3 | X \geq 1] &= \frac{\Pr[X \geq 3 \wedge X \geq 1]}{\Pr[X \geq 1]} \\
&= \frac{\Pr[X \geq 3]}{\Pr[X \geq 1]} \\
&= \frac{\int_3^\infty e^{-x}\,dx}{\int_1^\infty e^{-x}\,dx} \\
&= \frac{e^{-3}}{e^{-1}} = e^{-2}
\end{aligned}$$

2 Convergence of Random Variables

2.1

Let U be a uniform $[0, 1]$ random variable. For $k \geq 1$, let $X_k = 1$ if $U \geq 1 - \frac{1}{k}$ and 0 otherwise.

- As $n \to \infty$ does $X_n \to 0$ in distribution? Prove it.
- Does $X_n \to 0$ is probability? Prove it.
- Does $X_n \to 0$ almost surely? Prove it.

1. $X_n \to 0$ in distribution. Many ways to prove it but here's one. $\phi_0(t) = e^{i0} = 1$. $\phi_{X_n}(t) = \frac{1}{n}e^{it} + 1 - \frac{1}{n} \to 1$ as $n \to \infty$.
2. $X_n \to 0$ in probability. $\Pr[|X_n - 0| > \epsilon] = \frac{1}{n} \to 0$ as $n \to \infty$.
3. $X_n \to 0$ almost surely. With probability 1, $U < 1$. Therefor there is some N large enough, that $1 - 1/N > U$. In that case $X_n = 0$ for all $n \geq N$, and so $\lim_{n\to\infty} X_n = 0$. Since this happens with probability 1, $X_n \to 0$ almost surely.

   Another way to see this is the following: The sequence $X_1, X_2, X_3, \ldots$ will look like this: $1, 1, 1, 1, 1, 0, 0, 0, \ldots$ where $X_k = 1$ for all $k \leq \frac{1}{1-U}$ and 0 for all $k > \frac{1}{1-U}$. If $U \neq 1$, $\frac{1}{1-U}$ is finite, and the sequence therefor has the limit 0. $U \neq 1$ with probability 1, so $X_k \to 0$ almost surely.

3 Limit Theorems

3.1

Say $X_1, X_2, \ldots$ are iid with mean 0, variance 1. Prove that (without just quoting a theorem)

$$\frac{X_1 + \cdots + X_n}{n} \to 0 \text{ in probability}$$

Let $U_n = \frac{X_1 + \cdots + X_n}{n}$. By Chebyshev's Inequality,

$$\Pr[|U_n| > \epsilon] \leq \frac{var(U_n)}{\epsilon^2} = \frac{1}{n\epsilon^2} \to 0$$

3.2

Let X_n be Poisson with mean n. Prove that $\frac{X_n - n}{\sqrt{n}} \to N(0,1)$ in distribution. You may quote a theorem as part of your proof (but you can prove this other ways as well).

Two possible ways to show this:

1. X_n has the same distribution as $\sum_{i=1}^n Y_i$ where Y_i's are independent Poisson random variables with mean 1. The Central Limit Theorem says that

$$\frac{\sum_{i=1}^n (Y_i - 1)}{\sqrt{n}} \xrightarrow{D} N(0,1)$$

2. We can use characteristic functions:

   Fact 1:

$$\phi_{X_n}(t) = \sum_{k=0}^{\infty} \frac{e^{itk} n^k e^{-n}}{k!} = e^{n(e^{it} - 1}$$

   so

$$\phi_{(X_n - n)/\sqrt{n}}(t) = e^{-it\sqrt{n}} e^{n(e^{it/\sqrt{n}} - 1)}$$

   Now to find the limit as $n \to \infty$, we use a power series expansion of the inner-most exponential,

$$\phi_{(X_n - n)/\sqrt{n}}(t) = e^{-it\sqrt{n}} e^{n(it/\sqrt{n} - t^2/2n + o(1/n))} = e^{-t^2/2 + o(1)}$$

   and since $e^{-t^2/2}$ is the characteristic function of a $N(0,1)$ random variable, the proof is complete.