Math 6221: Homework 1

Due January 22

1

Let $\mathcal{A}$ be some collection (possibly infinite) of subsets of a set Ω. Prove that there is unique smallest σ-field containing $\mathcal{A}$. (We will call this $\sigma(\mathcal{A})$, the σ-field generated by $\mathcal{A}$),

2

Let $\Omega = \mathbb{Z}$. Let $\mathcal{F}$ be the family of all subsets of $\mathbb{Z}$ that are finite or whose complement is finite. Is $\mathcal{F}$ a σ-field?

3

Let X be a random variable with a continuous distribution function $F(t)$. Define a random variable $Y = F(X)$. What is the distribution of Y?

4

Let $\{f_n\}$ be a sequence of measurable functions. Prove that $g(x) = \limsup_{n\to\infty} f_n(x)$ is also measurable.

5

Let Z have a standard normal distribution. Prove that:

$$\Pr[Z > t] \le \frac{1}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2}$$

for $t > 0$. Find the best lower bound you can on $\Pr[Z > t]$ for large t.

6

[General Inclusion / Exclusion] Find a formula for

$$\Pr[A_1 \cup A_2 \cup \cdots \cup A_n]$$

in terms of probabilities of intersections of events.

7

Prove that if X and Y are independent random variables, then $\sigma(X)$ and $\sigma(Y)$ are independent σ-fields.

8

Toss a fair coin repeatedly. What is the probability that eventually you toss a head? Prove it.

9

Explain the Monty Hall problem.

10

What are necessary and sufficient conditions for

$$\Pr[A|B] = \Pr[B|A]$$

11

What are necessary and sufficient conditions for an event A to be independent of itself?

12

Let A and B be independent events. Prove that A^c and B^c are independent.

13

Show that a continuous function $f : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ is measurable, where $\mathcal{B}$ is the Borel sigma-field. Show that $\mathcal{B}$ is the smallest sigma-field on the input space that makes all continuous functions measurable.