Math 6221: Homework 3

Due February 19

1

Let S_n be a simple symmetric random walk.

- Find the asymptotics of $\Pr[S_n = 0]$ and $\Pr[S_n = n/5]$.
- Calculate $\mathbb{E}[S_1 | S_{100} = 20]$.
- Calculate $\mathbb{E}[S_{100} | S_1 = 1]$.
- What is $\mathbb{E}[S_1 | S_{100}]$?
- Find $\mathbb{E}[S_5 | S_{100}]$ and $\mathbb{E}[S_5 | S_{100} = 20]$.

2

Let S_n be a simple random walk with probability of a $+1$ step .6.

- Prove that with probability 1, $S_n \to \infty$.
- Prove that with probability 1, $S_n = 0$ only finitely many times.

3

Throw m balls uniformly and independently into n bins. Let Z_i be the number of balls in bin i.

- Show that $\frac{n}{m} Z_i$ converges in probability to 1.
- Let $m = n$. For the smallest $f(n)$ you can find, show that whp $|Z_1 - Z_2| \le f(n)$.
- For the smallest $g(n)$ you can find, show that $\max_i Z_i \le g(n)$ whp.
- Now let $m = cn$ for some constant c. Let E_n be the number of empty bins. Show that

$$\frac{E_n}{n} \to e^{-c}$$

  in probability as $n \to \infty$.

4

What's the probability a simple random walk hits 10 at step 40, without ever going below -2?

5

Consider a rv X with mean 3 and $\mathbb{E}e^X = 2$. Give the best bound you can on $\Pr[X > 7]$.

6

Draw a diagram showing the implications of the four types of convergence:

1. in distribution
2. in probability
3. almost sure
4. l_p

Give a proof in each direction there is an implication, and a counter example in each direction there is not an implication.

7

Consider the random graph $G(n, p)$, the graph on n vertices in which each potential edge is present independently with probability p. Let X be the number of triangles and Y the number of isolated vertices. Let $p = \frac{c}{n}$.

- Calculate the mean and variance of X and Y.
- Is there a 'weak law of large numbers' for X or Y? I.e. can you find a scaling factor $f(n)$ so that
$$\frac{X}{f(n)} \to 1$$
in probability? Same question for Y.

8

Prove that if $|X_n| < 20$ with probability 1, and $X_n \to X$ in probability, then $X_n \to X$ in l_2 and l_1.

9

Prove that if $X_n \to c$ in distribution, then $X_n \to c$ in probability.