Asymptotics

Will Perkins

January 22, 2013

Asymptotics

In many theorems and questions in probability theory, the perspective is *asymptotic*: there is some parameter n, and we are interested in characterizing behavior as n gets very large. The famous theorems in probability have this perspecitve: the Law of Large Numbers and the Central Limit Theorem.

We need some notation and technqiues to deal with asymptotics efficiently.

Asymptotic Equivalence

We write:

$$f(n) \sim g(n)$$

if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$

Examples:

1. $n^2 - 100n + 27 \sim n^2$
2. $\frac{n}{n - \log n} \sim 1$
3. $\binom{n}{7} \sim n^7/7!$

Big-Oh Notation

We write:

$$f(n) = O(g(n))$$

if there is some K so that

$$\limsup_{n\to\infty} \frac{f(n)}{g(n)} \leq K$$

In other words, there is some N so that for all $n \geq N$, $f(n) \leq Kg(n)$.

Examples:

1. $10n^2 + 100n = O(n^2)$
2. $100n = O(n^2)$
3. $\binom{n}{7} = O(n^7)$

Big-Theta Notation

We write:

$$f(n) = \Theta(g(n))$$

if $f(n) = O(g(n))$ and $g(n) = O(f(n))$. I.e. there is some $0 < c, K < \infty$ so that

$$c \leq \limsup_{n\to\infty} \frac{f(n)}{g(n)} \leq K$$

Examples:

1. $10n^2 + 100n = \Theta(n^2)$
2. $\binom{n}{7} = \Theta(n^7)$

Little-oh Notation

We write:

$$f(n) = o(g(n))$$

if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

In other words, for all $\epsilon > 0$, there is some N so that for all $n \geq N$, $f(n) \leq \epsilon g(n)$.

Examples:

1. $10n^2 + 100n = o(n^3)$
2. $100n = o(n^2)$
3. $\binom{n}{7} = o(n^8)$

Big and Little Omega Notation

We write:

$$f(n) = \Omega(g(n))$$

if there is some $c > 0$ so that for sufficiently large n,

$$f(n) \geq cg(n)$$

We write:

$$f(n) = \omega(g(n))$$

if

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$$

Exmaples:

1. $\binom{n}{7} = \Omega(n^7)$
2. $\binom{n}{7} = \omega(n^6)$

Asymptotics and Probability

Often we are concerned with 'typical' behavior as $n \to \infty$.

One definition of a typical event is that the probability tends to 1 as $n \to \infty$. I.e. $\Pr(A) = 1 - o(1)$ in little-oh notation.

The following are all equivalent ways of saying the same thing:

1. $\Pr(A) \to 1$ as $n \to \infty$
2. $\Pr(A) = 1 - o(1)$
3. A occurs 'with high probability' or 'whp'.

Stirling's Formula

Theorem (Stirling's Formula)

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

Power Series

There are a few power series that are helpful in finding asymptotics:

1. $e^x = 1 + x + x^2/2 + \cdots + x^k/k! + \ldots$
2. $\log(1 + x) = x - x/2 + x/3 - \ldots$
3. $\cosh x = 1 + x^2/2! + x^4/4! + \ldots$

Useful Limits

$$\left(1 + \frac{x}{n}\right)^n \to e^x$$

Example

What is the probability that a simple symmetric random walk $= 0$ at step n? Assume n is even. [Describe simple symmetric random walk]. This is the same as the probability a $Bin(n, 1/2) = 0$.
Exact:

$$\Pr[S_n = 0] = \binom{n}{n/2}(1/2)^n$$

Example

Asymptotics: use Stirling's Formula and cancel:

$$\binom{n}{n/2}(1/2)^n = \frac{n!2^{-n}}{(n/2)!(n/2)!}$$
$$\sim \frac{n^n e^{-n}\sqrt{2\pi n}2^{-n}}{(n/2)^n e^{-n}\pi n}$$
$$\sim \sqrt{\frac{2}{\pi n}}$$

An Exercise in Asymptotics

For constant k, we know that $\binom{n}{k} \sim n^k/k!$. Does that still hold if k depends on n?
For $k = k(n)$, find the asymptotics of:

$$\frac{\binom{n}{k}}{n^k/k!}$$

Find for:

- $k = o(n^{1/2})$
- $k = o(n^{2/3})$

Other asymptotics

All of the above definitions can be used with other parameters, besides $n \to \infty$. For example,

$$5x^2 + 3x \sim 3x$$

as $x \to 0$.