Characteristic Functions

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February 14, 2013

Characteristic Functions

Definition

The characteristic function of a random variable X is:

$$\phi_X(t) = \mathbb{E}e^{itX}$$

Properties of Characteristic Functions

Properties:

1. $\phi_X(0) = 1$
2. $|\phi_X(t)| \leq 1$ for all t
3. ϕ_X is uniformly continuous
4. $\phi_{aX+b}(t) = e^{itb}\phi_X(at)$
5. If X and Y are independent, $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$
6. If $\phi_X^{(k)}(0)$ exists, then $\mathbb{E}|X^k| < \infty$ if k even, $\mathbb{E}|X^{k-1}| < \infty$ if k odd.
7. If $\mathbb{E}|X^k| < \infty$, then $\phi_X^{(k)}(0) = i^k\mathbb{E}(X^k)$

Examples

- Bernoulli: $(1-p)+pe^{it}$
- Binomial $(1-p+pe^{it})^n$.
- Poisson: $e^{\lambda(e^{it}-1)}$
- Continuous uniform: $\frac{e^{itb}-e^{ita}}{it(b-a)}$
- Normal: $e^{-t^2/2}$

Inversion Formula

Theorem

Let μ be the distribution of a random variable X with characteristic function $\phi(t)$. Then

$$\mu(a,b) + \frac{1}{2}\mu(\{a,b\}) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\phi(t)\,dt$$

Corollary

Two random variables have the same distribution if and only if they have the same characteristic function.

Continuity Theorem

Theorem

Let $X_1, X_2, \ldots$ be a sequence of random variables with characteristic functions $\phi_n(t)$. Then

1. *If $X_n \Rightarrow X$, $\phi(t) = \lim \phi_n(t)$ exists and is the characteristic funcrion of X.*
2. *If $\phi(t) = \lim \phi_n(t)$ exists and is continuous at 0, then ϕ is characteristic function of a random variable X and $X_n \Rightarrow X$.*

Pointwise convergence of characteristic functions is equivalent to convergence in distribution.

Whis is continuity at 0 needed? Eg. $N(0, n)$..

Continuity Theorem

1) e^{itx} is a bounded continous function of x.

2) 2nd part says that it we don't have to check all bounded, continuous functions, just e^{itx}. For a proof see Durrett, 2.3

Example

Show that $Bin(n, \lambda/n) \Rightarrow Pois(\lambda)$