Expectation

Will Perkins

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Expectation

Definition

The *expectation* of a random variable X on a probability space $(\Omega, \mathcal{F}, P)$ is:

$$\mathbb{E}(X) = \int_{\Omega} X(\omega)\, dP(\omega)$$

Change of Variables

Often we think of a random variable without explicitly defining the probability space on which it lies. We can still compute its expectation using the formula:

$$\mathbb{E}(X) = \int_{\mathbb{R}} x \, dF_X(x)$$

where $dF_X(x)$ is the distribution of X; i.e. the measure on $\mathbb{R}$ induced by the random variable X, with the Borel σ-field of $\mathbb{R}$, and generated by

$$dF_X((-\infty, x]) = F(x)$$

Functions of X

Similarly we can define the expectation of any function of X. Say $g : \mathbb{R} \to \mathbb{R}$ is a measurable function. Then we define

$$\begin{aligned} \mathbb{E}(g(X)) &= \int_{\mathbb{R}} g(x)\, dF(x) \\ &= \int_{\Omega} g(X(\omega))\, dP(\omega) \end{aligned}$$

Properties of Expectation

Some basic properties of expectation that follow from the properties of abstract integration:

1. Linearity: $\mathbb{E}(aX + bY) = a\mathbb{E}X + b\mathbb{E}Y$.
2. Monotonicity: if $X \geq Y$ a.s., then $\mathbb{E}X \geq \mathbb{E}Y$.
3. Jensen's Inequality: Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then
$$f(\mathbb{E}X) \leq \mathbb{E}(f(X))$$

Examples

Let X be the indicator random variable of an event A.

Then $\mathbb{E}X = 1 \cdot \Pr(A) + 0 = \Pr(A)$.

Examples

Poisson Distribution: Let $X \sim \mathit{Pois}(\lambda)$. $\mathbb{E}X = ?$

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Continuous RV's: Let $X \sim \mathit{Uniform}[0,1]$.
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Binomial Distribution: Let $X \sim Bin(n, p)$. $\mathbb{E}X = ?$

Use Linearity of Expectation.

The power of linearity is that *dependencies don't matter*.

Expectation of Counting Random Variables

Counting random variables are somewhat special: 'The number of...'. A binomial is a simple example, the number of heads in n flips. But there are many more complicated examples:

1. The number of times a random walk hits 0 in n steps.
2. The number of integer solutions of a random set of linear inequalities.
3. The number of neighbors of a vertex in a random graph.

and so on.

Expectation of Counting Random Variables

Here's a useful framework for computing expectations of counting random variables.

1. Write X as a sum of indicator RV's: $X = X_1 + X_2 + \dots X_n$, where X_i is either 1 or 0. Each indicator rv should correspond to one of the possible things being counted; e.g., $X_i = 1$ if the ith flip is a head.
2. Calculated $\mathbb{E}X_i = \Pr[X_i = 1]$
3. $\mathbb{E}X = \sum_i \mathbb{E}X_i$

The nice thing is that it doesn't matter whether or not the X_i's are independent!

Examples

A quick detour: The Erdős-Rényi *Random Graph* is a distribution over graphs on n vertices in which each of the $\binom{n}{2}$ potential edges is present independently with probability p.

Questions:

1. What is the expected degree of a given vertex?
2. What is the expected number of isolated vertices? (vertices with dergree 0)