Martingale Convergence

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Martgingales

Recall that a sequence of random variables X_n is a Martingale with respect to a filtration $\mathcal{F}_1 \subseteq \mathcal{F}_2 \cdots$ if:

1. $\mathbb{E}|X_n| < \infty$
2. $\mathbb{E}[X_n|\mathcal{F}_{n-1}] = X_{n-1}$

Sub- and Super-martingales

A submartingale is a sequence of rv's so that

$$\mathbb{E}[X_n|\mathcal{F}_{n-1}] \geq X_{n-1}$$

A supermartingale is a sequence of rv's so that

$$\mathbb{E}[X_n|\mathcal{F}_{n-1}] \leq X_{n-1}$$

A martingale is both a submartingale and a supermartingale.

Stopping Times

An integer-valued random variable N is a *stopping time* with respect to the filtration $\mathcal{F}_n$ if the event

$$\{N = n\} \in \mathcal{F}_n$$

Examples:

- Let N be the first time $S_n \geq 10$.
- $N = 4$ (a constant)
- Not a stopping time: N is the *last* time $S_n = 0$.

Stopping Times

Theorem

If N is a stopping time and S_n is a martingale, then $S_{n \wedge N}$ is a martingale.

Proof:

Predictable Sequences

Definition

A sequence $\{H_n\}$ is a predictable sequence with respect to a filtration $\mathcal{F}_n$ if

$$H_n \in F_{n-1}$$

for all n.

Back to the gambling, think of H_n as the amount the gambler bets on the nth roll of the dice. It can depend on all that has happened up til step n, but it cannot depend on the nth flip or anything in the future.

Predicatable seqeuences and Martingales are the building blocks of integrals in stochastic calculus.

Discrete Stochastic Integration

Define

$$(H \cdot S)_n = \sum_{j=1}^{n} H_n(S_n - S_{n-1})$$

(In the gambling example, this is the amount of money the gambler has won by time n). Note that $(H \cdot S)_n$ is a random variable.

Fact: If S_n is a Martingale and H_n is a predictable sequence, then $(H \cdot S)_n$ is a Martingale.
Proof: ?

Doob's Upcrossing Inequality

Let S_n be a stochastic process and pick two numbers $a < b$. The number of upcrossings $U(a, b)$ of S_n is the number of times the process goes from being at or below a to at or above b.
[Draw an example]

Between any two upcrossings is a down crossing, and vice-versa.

Lemma

If for every $a < b$, $U(a, b) < \infty$, then S_n converges to a limit.

Doob's Upcrossing Inequality

Counting Upcrossings with Stopping Times:
Define two sets of interlaced stopping times:

$$s_k = \inf\{n > t_{k-1} : S_n \leq a\}$$

$$t_k = \inf\{n > s_k : S_n \geq b\}$$

and set $t_0 = 0$.

Then the number of upcrossings up to step N is the largest k so that $t_k \leq N$.

Doob's Upcrossing Inequality

Let

$$Y_n = \mathbf{1}_{s_k \le n \le t_k \text{ for some } k}$$

$$\begin{aligned}
\mathbb{E}(Y \cdot S)_n &= \sum_{j=1}^{\infty} (S_{t_j \wedge n} - S_{s_j \wedge n}) \\
&= \sum_{j=1}^{K} (S_{t_j} - S_{s_j}) + (X_n - X_{s_K}) \\
&\ge K(b-a) - \max(a - X_n, 0)
\end{aligned}$$

where K is the number of (a, b) upcrossings up to time n.

Doob's Upcrossing Inequality

Theorem (Upcrossing Inequality)

$$(b-a)\mathbb{E}[U(a,b)] \leq \sup_n \mathbb{E}[(a-X_n)^+]$$

Almost Sure Convergence

Theorem

Let X_n be a martingale with $\mathbb{E}X_n^+ < M$ for all n. Then there exists a random variable X with $\mathbb{E}|X| < \infty$ so that

$$X_n \to X \text{ almost surely}$$

Proof: Use the Upcrossing Inequality with $-X_n$ (also a Martingale) to see that for any (a, b),

$$\mathbb{E}U(a, b) < \infty$$

And so with probability 1, the number of a, b upcrossings is finite. Now take a countable union over all rational $a < b$ to get the conclusion.

For the expectation, use Fatou's Lemma.

Almost Sure Convergence

Corollary

Let $X_n \geq 0$ *be a Martingale. Then* $X_n \to X$ *a.s.*

Example 1

Let M_n be a SSRW stopped when $S_n = 10$. Show that M_n converges almost surely and to what.
What is $\mathbb{E}M_n$?

Example 2

Consider a Galton-Watson branching process with mean offsprsing size μ. Let $W_n = \frac{Z_n}{\mu^n}$. Show that $W_n \to W$ almost surely.