Martingales

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A Betting System

Here's a strategy for making money (a dollar) at a casino:

- Bet \$1 on Red at the Roulette table.
- If you win, go home with \$1 profit.
- If you lose, bet \$2 on the next roll.
- Repeat.
- What could go wrong?

This strategy is called "The Martingale" and it is slightly related to a mathematical object called a martingale.

Conditional Expectation

Early in the course we defined the conditional expectation of random variables given an event or another random variable. Now we will generalize those definitions.

Let X be a random variable on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{F}_1 \subseteq \mathcal{F}$. Then we define the conditional expectation of X with respect to $\mathcal{F}_1$

$$\mathbb{E}[X|\mathcal{F}_1]$$

as a random variable Y so that:

1. Y is $\mathcal{F}_1$ measurable.
2. For any $A \in \mathcal{F}_1$, $\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[Y\mathbf{1}_A]$

Properties of Conditional Expectation

1. Linearity:

$$\mathbb{E}[aX + Y|\mathcal{F}_1] = a\mathbb{E}[X|\mathcal{F}_1] + \mathbb{E}[Y|\mathcal{F}_1]$$

2. Expectations of expectations:

$$\mathbb{E}[\mathbb{E}[X|\mathcal{F}_1]] = \mathbb{E}[X]$$

3. Pulling a $\mathcal{F}_1$-measurable function out: If Y is $\mathcal{F}_1$-measurable, then

$$\mathbb{E}[XY|\mathcal{F}_1] = Y\mathbb{E}[X|\mathcal{F}_1]$$

4. Tower property: if $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then

$$\mathbb{E}[\mathbb{E}[X|\mathcal{F}_1]|\mathcal{F}_2] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_2]|\mathcal{F}_1] = \mathbb{E}[X|\mathcal{F}_1]$$

Properties of Conditional Expectation

Propery (2) is a special case of property (4), since

$$\mathbb{E}[X] = \mathbb{E}[X|\mathcal{F}_0]$$

where $\mathcal{F}_0$ is the smallest possible σ-field, $\{\Omega, \emptyset\}$.

Properties of Conditional Expectation

Example of property (3):
Let S_n be a SSRW, and let $\mathcal{F}_n = \sigma(S_1, \ldots S_n)$, where X_i's are the ± 1 increments.

Calculate $\mathbb{E}[S_n^2|\mathcal{F}_k]$ for $k < n$:

$$
\begin{aligned}
\mathbb{E}[S_n^2|\mathcal{F}_k] &= \mathbb{E}[(S_k + (S_n - S_k))^2|\mathcal{F}_k] \\
&= \mathbb{E}[S_k^2 + 2S_k(S_n - S_k) + (S_n - S_k)^2|\mathcal{F}_k] \\
&= S_k^2 + 2S_k\mathbb{E}[S_n - S_k|\mathcal{F}_k] + \mathbb{E}[(S_n - S_k)^{|}\mathcal{F}_k] \\
&= S_k^2 + 0 + n - k = S_k^2 + n - k
\end{aligned}
$$

For this definition to make sense we need to prove two things:

1. Such a Y exists.
2. It is unique.

Uniqueness: Let Z be another random variable that satisfies 1) and 2). Show that $\Pr[Z - Y > \epsilon] = 0$ for any $\epsilon > 0$. Show that this implies that $Z = Y$ a.s.

Existence

We start with some real analysis:

Definition

A measure Q is said to be absolutely continuous with respect to a measure P (on the same measurable space) if

$$P(A) = 0 \Rightarrow Q(A) = 0.$$

We write $Q << P$ in this case.

Example: The uniform distribution on $[0, 1]$ is absolutely continuous with respect to the Gaussian measure on $\mathbb{R}$, but not vice-versa.

Radon-Nikodym Theorem

We will need the following classical theorem:

Theorem (Radon-Nikodym)

Let P and Q be measures on $(\Omega, \mathcal{F})$ so that $P(\Omega), Q(\Omega) < \infty$. Then if $Q << P$, there exists an $\mathcal{F}$ measurable function f so that for all $A \in \mathcal{F}$,

$$\int_A f \, dP = Q(A)$$

f is called the Radon-Nikodym derivative and is written

$$f = \frac{dQ}{dP}$$

Existence of Conditional Expectation

Let $X \geq 0$ be a random variable on $(\Omega, \mathcal{F}, P)$. For $A \in \mathcal{F}$, define:

$$Q(A) = \int_A X \, dp$$

1. Q is a measure
2. $Q << P$

Now let $Y = \frac{dQ}{dP}$. Show that $Y = \mathbb{E}[X|\mathcal{F}]$!

Conditional Expectation

Show that the above definition generalizes our previous definitions of conditional expectation given and event or a random variable.

A Filtration

Definition

A filtration is a sequence of sigma-fields on the same measurable space so that

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots$$

Example: Let S_n be a simple random walk, and define

$$\mathcal{F}_n = \sigma(S_1, \ldots S_n)$$

Think of a Filtration as measuring information revealed during a stochastic process.

Martingales

Definition

A Martingale is a stochastic process S_n equipped with a sigma-field $\mathcal{F}_n$ so that $\mathbb{E}[|S_n|] < \infty$ and

$$\mathbb{E}[S_n|\mathcal{F}_{n-1}] = S_{n-1}$$

Exercise: Prove that simple symmetric random walk with the natural filtration is a Martingale.

Martingales

Martingales are a generalization of sums of independent random variables. The increments need not be independent, but they have the martingale property (mean 0 conditioned on the current state).

An example with dependent increments: Galton-Watson Branching process. Show that Z_n with its natural filtration is a Martingale.

A Gambling Martingale

- Let S_n be a gambler's 'fortune' at time n. Say $S_0 = 10$.
- At each step the gambler can place a bet, call it b_n. The bet must not be more than the current fortune.
- With probability $1/2$ the gambler wins b_n, with probability $1/2$ the gambler loses b_n.
- The bet can depend on anything in the past, but not the future.

Show that for any betting strategy S_n is a Martingale.

Expectations

Say S_n is a Martingale with $S_0 = a$.
What is $\mathbb{E}S_n$?

Constructing Martingales

Let X be a random variable on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}$ be a filtration.
Define

$$M_n = \mathbb{E}[X|\mathcal{F}_n]$$

Prove that $(M_n, \mathcal{F}_n)$ is a Maringale.

This type of Martingale is called Doob's Margingale.

Examples of Doob's Martingales

1. Let S_n be a simple random walk with bias p. Construct a Doob's martingale from $X = S_n$. How about S_n^2 or S_n^3?
2. Let X be the number of triangles in $G(n, p)$. Construct a filtration and a martingale that converges to X. What are natural filtrations on the probability space defined by $G(n, p)$?
3. Can you construct a Doob's martingale associated to a branching process?