Poisson Point Processes

Will Perkins

April 23, 2013

The Poisson Process

Say you run a website or a bank. How woul you model the arrival of customers to your site?

Continuous time process, integer valued. What properties should the process have?

Properties

1. The numbers of customers arriving in disjoint time intervals are independent.
2. The number of customers arriving in $[t_1, t_2]$ depends only on $t_2 - t_1$. (Can be relaxed)
3. The probability that one customer arrives in $[t, t+\epsilon]$ is $\epsilon\lambda + o(\epsilon)$.
4. The probability that at least two customers arrive in $[t, t+\epsilon]$ is $o(\epsilon)$.

The Poisson Process

Theorem

If a process $N(t_1, t_2)$ satisfies the above properties, then $N(0, t)$ has a Poisson distribution with mean λt.

Such a process is called a Poisson process.

Proof:

Other Properties

1. Conditioning on the number of arrivals in $[0, T]$, how are the arrival times distributed?
2. What is the distribution of the time between arrival k and $k+1$?
3. Does this process have the continuous-time Markov property?

Proofs:

Constructing a Poisson Process

We can construct a Poisson process using a sequence of iid random variables.
Let $X_1, X_2, \ldots$ be iid Exponential rv's with mean $1/\lambda$. Then let

$$N(0,t) = \inf\{k : \sum_{i=1}^{k+1} X_i \geq t$$

- Show that this is a Poisson process with mean λ.
- What would happend if we chose a different distribution for the X_i's?

Inhomogeneous Poisson Process

Let $f(t)$ be a non-negative, integrable function. Then we can define an inhomogeneous Poisson process with intensity measure $f(t)$ as follows:

1. The number of arrivals in disjoint intervals are independent.
2. The number of arrivals in $[t_1, t_2]$ has a Poisson distribution with mean $\mu = \int_{t_1}^{t_2} f(t)\,dt$.

Spatial Poisson Process

We can think of the Poisson process as a random *measure* on $\mathbb{R}$. This is an infinite point measure, but assigns finite measure to any bounded subset of $\mathbb{R}$.

Can we generalize to $\mathbb{R}^2$ or $\mathbb{R}^3$?

Spatial Poisson Process

Define a random measure μ on $\mathbb{R}^d$ (with the Borel σ-field) with the following properties:

1. If $A \cap B = \emptyset$, then $\mu(A)$ and $\mu(B)$ are independent.
2. $\mu(A)$ has a Poisson distribution with mean $\lambda m(A)$ where $m(A)$ is the Lebesgue measure (area).

This is a spatial Poisson process with intensity λ. Similarly, we can define an inhomogeneous spatial Poisson process with intensity measure $f : \mathbb{R}^d \to [0, \infty)$.

Spatial Poisson Process

Exercise 1: Show that the union of two independent Poisson point processes is itself a Poisson point process.

Exercise 2: Take a Poisson point process on $\mathbb{R}^d$ and then independently color each point red with probability p and green otherwise. Show that the red points and the green points form independent Poisson point processes.

Exercise 3: Conditioned on the number of points in a set A, find the distribution of the positions of the points in A.