# On the zeros and approximation of the Ising partition function

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# Partition function

Let $f : \{-1, 1\}^n \longrightarrow \mathbb{R}, \mathbb{C}$ be a function.

- We want to efficiently compute (approximate)

$$\sum_{x \in \{-1,1\}^n} e^{f(x)}.$$

A closely related question:

- When

$$\sum_{x \in \{-1,1\}^n} e^{f(x)} \neq 0?$$

# Partition function

We are interested in the cases when $f$ is

$$\text{quadratic:} \quad f(x) = \sum_{1 \leq i < j \leq n} a_{ij} \xi_i \xi_j + \sum_{i=1}^n b_i \xi_i$$

$$\text{or cubic:} \quad f(x) = \sum_{1 \leq i < j < k \leq n} c_{ijk} \xi_i \xi_j \xi_k + \sum_{1 \leq i < j \leq n} a_{ij} \xi_i \xi_j + \sum_{i=1}^n b_i \xi_i.$$

for $x = (\xi_1, \ldots, \xi_n)$.
Remark: If

$$f(x) = \alpha + \sum_{i=1}^n b_i \xi_i$$

is affine, then

$$\sum_{x \in \{-1,1\}^n} e^{f(x)} = e^{\alpha} \prod_{i=1}^n \left( e^{b_i} + e^{-b_i} \right)$$

and everything is easy (need $e^{b_i}$ as an input).

# Quadratic $f$

**Theorem (Theorem 1)**

*Suppose that $f$ is quadratic and that for some $0 < \delta < 1$, we have*

$$\sum_{j:\ j \neq i} |\Re\, a_{ij}| \ \leq \ 1 - \delta, \quad \sum_{j:\ j \neq i} |\Im\, a_{ij}| \ \leq \ \frac{\delta^2}{10} \quad \text{and} \quad |\Im\, b_i| \ \leq \ \frac{\delta^2}{10}$$

*for $i = 1, \ldots, n$. Then*

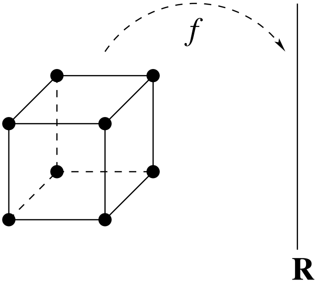
$$\sum_{x \in \{-1,1\}^n} e^{f(x)} \neq 0.$$

Corollary: By interpolation, $\sum_{x \in \{-1,1\}^n} e^{f(x)}$ can be approximated within relative error $0 < \epsilon < 1$ in quasi-polynomial $n^{O(\ln n - \ln \epsilon)}$ time, provided $a_{ij}, b_i \in \mathbb{R}$ and

$$\sum_{j:\ j \neq i} |a_{ij}| \ \leq \ 1 - \delta \quad \text{for} \quad i = 1, \ldots, n.$$

# Quadratic $f$

Again, we need $e^{b_i}$ as an input. Geometrically, everything is easy if the Lipschitz constant of the non-linear part of $f$ is strictly less than 1, with respect to the $\ell^1$ metric on $\{-1, 1\}^n$.



# Cubic $f$

**Theorem (Theorem 2)**

*Suppose that $f$ is cubic and that for some $0 < \delta < 1/2$, we have*

$$\sum_{j,k:\ j,k \neq i} |\Re\, c_{ijk}| + \sum_{j:\ j \neq i} |\Re\, a_{ij}| \ \leq\ 1 - \delta$$

$$\sum_{j,k:\ j,k \neq i} |\Im\, c_{ijk}| + \sum_{j:\ j \neq i} |\Im\, a_{ij}| \ \leq\ \frac{\delta^2}{10} \quad \text{and}$$

$$|\Im\, b_i| \ \leq\ \frac{\delta^2}{10} \quad \text{for} \quad i = 1, \ldots, n.$$

*Then*

$$\sum_{x \in \{-1,1\}^n} e^{f(x)} \neq 0.$$

# Cubic $f$

Corollary: By interpolation, $\sum_{x \in \{-1,1\}^n} e^{f(x)}$ can be approximated within relative error $0 < \epsilon < 1$ in quasi-polynomial $n^{O(\ln n - \ln \epsilon)}$ time, provided $c_{ijk}, a_{ij}, b_i \in \mathbb{R}$ and

$$\sum_{j,k:\ j,k \neq i} |c_{ijk}| + \sum_{j:\ j \neq i} |a_{ij}| \ \leq\ 1 - \delta \quad \text{for} \quad i = 1, \ldots, n.$$

Again, we need $e^{b_i}$ as an input.

# General $f$

Generally, if $f : \{-1, 1\}^n \longrightarrow \mathbb{R}, \mathbb{C}$ is a polynomial of degree $d$,

$$f(x) = \sum_{\substack{I \subset \{1,\ldots,n\} \\ |I| \leq d}} a_I \prod_{i \in I} \xi_i \quad \text{where} \quad x = (\xi_1, \ldots, \xi_n),$$

then similar results can be obtained assuming that

$$\sum_{I:\ i \in I} |a_I| \ \leq \ \frac{\gamma}{\sqrt{d}},$$

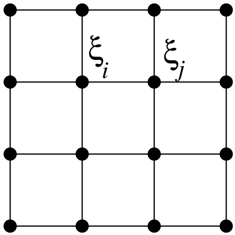
where $\gamma > 0$ is an absolute constant.
In the cases of $d = 2$ and $d = 3$ we get asymptotically optimal bounds.

# Ising model on graphs

Let $G = (V, E)$ be a graph with vertices $V = \{1, \ldots, n\}$ and edges $E$. For a real $a$, let us choose

$$a_{ij} = \begin{cases} a & \text{if} \quad \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

and suppose that $b_i = b$ for some $b \in \mathbb{C}$ and $i = 1, \ldots, n$.



# Ising model on graphs: zeros

Let $\Delta \geq 3$ be the largest degree of a vertex of $G$ and suppose that either

$$a = \frac{1}{2} \ln \frac{\Delta}{\Delta - 2} \quad \text{(ferromagnetic interactions)} \quad \text{or}$$
$$a = \frac{1}{2} \ln \frac{\Delta - 2}{\Delta} \quad \text{(antiferromagnetic interactions)}.$$

As $G$ ranges over the graphs with largest degree $\Delta$, the zeros of the functions

$$b \longmapsto \sum_{x \in \{-1,1\}^n} e^{f(x)}$$

can get arbitrarily close to 0: [Barata and Goldbaum, 1991], [Barata and Marchetti, 1997], [Bencs, Buys, Guerini and Peters 2019], [Peters and Regts, 2020].

Now, in this case

$$\sum_{j:\ j \neq i} |\Re\, a_{ij}| \ \leq \ \frac{\Delta}{2} \ln \frac{\Delta}{\Delta - 2} \longrightarrow 1 \quad \text{as} \quad \Delta \longrightarrow \infty,$$

and hence "1" in Theorems 1 and 2 is optimal.
Let us choose any

$$a \ < \ \frac{1}{2} \ln \frac{\Delta - 2}{\Delta} \quad \text{(antiferromagnetic interactions)}.$$

The problem of approximation of

$$\sum_{x \in \{-1,1\}^n} e^{f(x)}$$

is NP-hard on graphs of the largest degree $\Delta \geq 3$ under randomized reductions [Sly and Sun, 2014], [Galanis, Štefankovič and Vigoda, 2016]. Hence "1" is optimal in the corollaries.

# The interpolation lemma

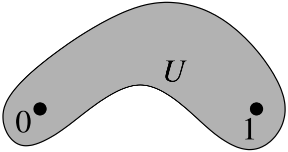
## Lemma

*Let $U \subset \mathbb{C}$ be a connected open set containing 0 and 1. Then there is a constant $\gamma = \gamma(U) > 0$ such that the following holds: If*

$$g(z) = \sum_{k=0}^{n} c_k z^k, \quad n \geq 2$$

*is a polynomial such that $g(z) \neq 0$ for all $z \in U$ then, for any $0 < \epsilon < 1$, the value of $g(1)$, up to relative error $\epsilon$, is determined by the coefficients $c_k$ with $k \leq \gamma (\ln n - \ln \epsilon)$ and can be computed in $n^{O(1)}$ time from those coefficients.*

Remark: We say that $w_1 \neq 0$ approximates $w_2 \neq 0$ within relative error $\epsilon$ if we can write $w_1 = e^{z_1}$ and $w_2 = e^{z_2}$ with $|z_1 - z_2| \leq \epsilon$.




If

$$g(z)=\sum_{k=0}^{n} c_k z^k$$

and $g(z) \neq 0$ in an open connected set containing 0 and 1, then, up to relative error $0<\epsilon<1$, the value of $g(1)$ is determined by only $O(\ln n - \ln \epsilon)$ lowest coefficients of $g$.

Roughly,

$$\gamma(U) \sim \frac{1}{\beta-1},$$

where $\beta>1$ is the largest radius of the disc $\mathbb{D}=\{z: \ |z|<\beta\}$ with a holomorphic map $\phi: \mathbb{D} \longrightarrow U$ satisfying $\phi(0)=0$ and $\phi(1)=1$.

# Zero freeness $\Longrightarrow$ approximation

Given

$$f(x) = \sum_{1 \leq i < j \leq n} a_{ij} \xi_i \xi_j + \sum_{i=1}^n b_i \xi_i \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n),$$

we write

$$e^{f(x)} = \exp\left\{ - \sum_{1 \leq i < j \leq n} a_{ij} \right\} \left( \prod_{1 \leq i < j \leq n} e^{a_{ij}(\xi_1 \xi_j + 1)} \right) \left( \prod_{i=1}^n e^{b_i \xi_i} \right)$$

$$= \exp\left\{ - \sum_{1 \leq i < j \leq n} a_{ij} \right\} \left( \prod_{1 \leq i < j \leq n} (1 + c_{ij})^{N(\xi_i \xi_j + 1)} \right) \left( \prod_{i=1}^n e^{b_i \xi_i} \right)$$

where $c_{ij} = e^{a_{ij}/N} - 1$ and $N = n^2$ and apply the Interpolation Lemma to the polynomial

$$g(z) = \sum_{x \in \{-1,1\}^n} \prod_{1 \leq i < j \leq n} (1 + z c_{ij})^{N(\xi_i \xi_j + 1)} \left( \prod_{i=1}^n e^{b_i \xi_i} \right).$$

# The idea of the proofs of Theorems 1 and 2

A *face* $F$ of the cube $\{-1, 1\}^n$ consists of the points $x = (\xi_1, \ldots, \xi_n)$, where

$$\xi_i = 1 \quad \text{for} \quad i \in I_+ \quad \text{and} \quad \xi_i = -1 \quad \text{for} \quad i \in I_-$$

while the remaining coordinates $\xi_i : \ i \notin I_+ \cup I_-$ are free to take either value.

We prove by induction on the dim $F$ that if $F_+ \subset F$ and $F_- \subset F$ are the faces of $F$ obtained by fixing a free coordinate to 1 and $-1$ respectively, then

$$s_+ = \sum_{x \in F_+} e^{f(x)} \neq 0, \quad s_- = \sum_{x \in F_-} e^{f(x)} \neq 0$$

and the angle between $s_+$ and $s_-$ (as vectors in $\mathbb{R}^2 = \mathbb{C}$) does not exceed some $\theta = \theta(\delta) > 0$.

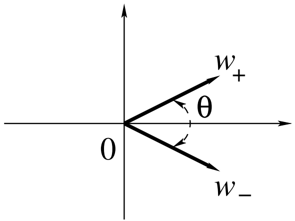
# A bit of geometry, $d = 2$

For $d = 2$, the proof uses the following geometric lemma.

**Lemma (Lemma 1)**

*Let $w_+, w_- \in \mathbb{C} \setminus \{0\}$ be numbers such that the angle between $w_+$ and $w_-$ does not exceed some $0 \leq \theta < \pi$. Then*

$$\left| \Im \frac{w_+ - w_-}{w_+ + w_-} \right| \leq \tan \frac{\theta}{2}.$$



# A bit of geometry, $d = 3$

For $d = 3$, the proof uses the following geometric lemma.

**Lemma (Lemma 2)**

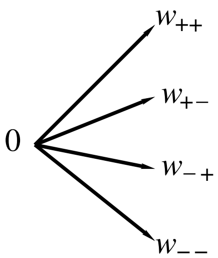
*Let* $w_{++}, w_{+-}, w_{-+}, w_{--} \in \mathbb{C} \setminus \{0\}$ *be numbers such that the angles*

*between* $w_{++}$ *and* $w_{+-}$
*between* $w_{++}$ *and* $w_{-+}$
*between* $w_{--}$ *and* $w_{+-}$
*between* $w_{--}$ *and* $w_{-+}$

*do not exceed* $\theta$ *for some* $0 \leq \theta < \pi/2$. *Then*

$$\left| \Im \frac{w_{++} - w_{-+} - w_{+-} + w_{--}}{w_{++} + w_{-+} + w_{+-} + w_{--}} \right| \leq \tan \frac{\theta}{2}.$$

# A bit of geometry, $d = 3$




Reduces to Lemma 1 if we let

$$w_+ = w_{++} + w_{--} \quad \text{and} \quad w_- = w_{-+} + w_{+-}.$$

The obvious extension to eight vectors

$$w_{+++}, w_{++-}, w_{+-+}, w_{-++}, w_{+--}, w_{-+-}, w_{--+}, w_{---}$$

fails (it would have taken care of $d = 4$).