MATH 215: Introduction to Advanced Mathematics

Review sheet for Midterm Test 1

Definitions to know

(1) Logical connectives: and, or, not, implies
(2) a divides b
(3) Summation and product notation: $\sum_{j=1}^{n} f(j)$ and $\prod_{j=1}^{n} f(j)$.
(4) Conditional definition of a set
(5) Intervals on the real line, e.g. (a, b), $[a, \infty)$, etc.
(6) Subset, strict subset: $A \subseteq B$, $A \subset B$.
(7) Equality of sets
(8) Empty set
(9) Power set $\mathcal{P}(S)$
(10) Union, intersection, complement of sets
(11) Difference of sets: $A - B$
(12) Cartesian product of sets
(13) Existential and universal quantifiers
(14) Function, domain, codomain, image
(15) Sequence $f(n)$
(16) Null sequence
(17) Composition of functions
(18) Injective
(19) Surjective
(20) Bijective
(21) Inverse function / invertible function

Techniques to know

(1) **Write a clear, correct, and concise proof!**
(2) Truth tables
(3) Implication arrows
(4) Direct proof
(5) Backwards proof
(6) Proof by cases
(7) Negate a statement with quantifiers
(8) Proof by contrapostitive
(9) Proof by contradiction
(10) Proof by induction
(11) Depict sets using a Venn diagram

Tips for writing good proofs

(1) Write in full sentences.
(2) Make it clear where the proof begins and ends.
(3) Use words with precision:
  - given
  - suppose
  - if
  - implies
  - then
(4) Make sure implications (and implication arrows) go the right way
(5) Make it clear where you use the assumptions of the statement

Practice problems

(1) Write the negation of the following statements.
  (a) For all integers n, n^3 is an odd number.

(b) There exist real numbers x, y, z so that $x^3 = 2y^2 - z^4$.
(c) For all even integers n, n^3 is odd implies $n = 7$.

(2) Prove that $\sum_{j=0}^{n} r^j = \frac{1-x^{n+1}}{1-x}$ for all positive integers n and all real $x \neq 0$.
(3) Prove by induction that $\sum_{j=1}^{n} 2^j = 2^{n+1} - 2$ for all positive integers n.
(4) Let $A = \{n \in \mathbb{Z} | -10 \leq n \leq 10\}$ and $B = \{n \in \mathbb{Z} | n^2 \geq 10\}$.
   (a) What is $A \cap B$?
   (b) What is $A - B$?
   (c) Is $A \subseteq B$?
   (d) Prove: $\forall a \in A, \exists b \in B, a + b < 0$.
   (e) Prove: $\exists a \in A, \forall b \in B, |a| < |b|$.
(5) Prove of disprove the following:
   (a) For all integers n, $n^3 + 3$ is divisible by 3.
   (b) For x, y real numbers, $x^2 > y^2$ implies $x > y$.
   (c) For real x, $x^2 \geq 16$ implies $x \geq 4$ or $x \leq -2$.
   (d) $\exists x \in (0, 10), \forall y \in (5, 10), x > y$.
(6) Let $f(n)$ be the sequence defined by $f(n) = e^{-n}$. Prove that $f(n)$ is a null sequence.
(7) For each function listed below, determine its image and whether the functions is injective, surjective, or bijective.
   - $f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = x^2 - 1$.
   - $f_2 : \mathbb{R}^{\geq} \to \mathbb{R}^{\geq}$ defined by $f_2(x) = x^2 + 1$.
   - $f_3 : (0, 1) \to (0, 2)$ defined by $f_3(x) = x + 1$.
   - $f_4 : (0, \infty) \to (0, \infty)$ defined by $f_4(x) = 1/x$.
(8) Prove or disprove: if $f : \mathbb{R} \to \mathbb{R}$ is injective and $g : \mathbb{R} \to \mathbb{R}$ is surjective then $f \circ g : \mathbb{R} \to \mathbb{R}$ is bijective.