Metastability for the dilute Curie–Weiss model with Glauber dynamics

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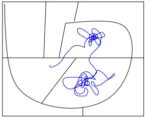
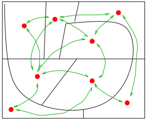
What is metastability?

Metastability is a phenomenon where a system, under the influence of a stochastic dynamics, moves between different regions of its state space on different time scales.





Fast time scale:
quasi-equilibrium within single subregion

Slow time scale:
transitions between different subregions

Monographs:

- Olivieri and Vares 2005
- Bovier and den Hollander 2015

The randomly dilute Curie–Weiss model

The RDCW model is a classical model of a disordered ferromagnet.
Ising spin model with N spins
Configuration space $\mathcal{S}_N = \{-1,+1\}^N$
Configuration $\sigma = (\sigma_i)_{i\in[N]} \in \mathcal{S}_N$, $\sigma_i \in \{-1,+1\}$
$[N] = \{1,2,\ldots,N\}$, $h > 0$ constant magnetic field.
Hamiltonian in the **randomly dilute Curie–Weiss model** (RDCW)

$$H_N(\sigma) = -\frac{1}{Np} \sum_{1\le i<j\le N} J_{ij}\sigma_i\sigma_j - h\sum_{i\in[N]}\sigma_i$$

where $\{J_{ij}\}_{i,j\in[N]}$ is a sequence of i.i.d. random variables such that $J_{ij} = J_{ji}$ and $\mathbb{E}(J_{ij}) = p \in (0,1)$ constant [e.g. $J_{ij} \sim$ Ber(p)]

Hamiltonian in the standard **Curie–Weiss model** (CW)

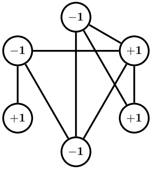
$$H_N^{\text{CW}}(\sigma) = -\frac{1}{N} \sum_{1\le i<j\le N} \sigma_i\sigma_j - h\sum_{i\in[N]}\sigma_i = \mathbb{E}(H_N(\sigma))$$

Graphical representation of configurations

Define the **interaction graph** $G = ([N], E) : (i,j) \notin E \iff J_{ij} = 0$

$$H_N(\sigma) = -\frac{1}{Np} \sum_{1 \le i < j \le N} J_{ij} \sigma_i \sigma_j - h \sum_{i \in [N]} \sigma_i$$

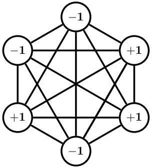
$$= -\frac{1}{Np} \sum_{\{i,j\} \in E} \sigma_i \sigma_j - h \sum_{i \in [N]} \sigma_i$$



We take $J_{ij} \sim \text{Ber}(p)$, $p \in (0,1) \implies G$ is an **Erdős–Rényi random graph** with fixed edge probability p

Standard Curie–Weiss model $\implies G$ is a complete graph

$$H_N^{\text{CW}}(\sigma) = -\frac{1}{N} \sum_{1 \le i < j \le N} \sigma_i \sigma_j - h \sum_{i \in [N]} \sigma_i$$



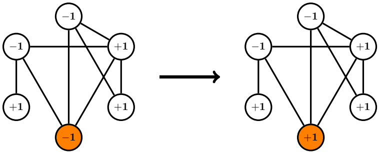
The Glauber dynamics

At equilibrium we define the Gibbs measure, $\sigma \in \mathcal{S}_N$,

$$\mu_{N,\beta}(\sigma) = \frac{e^{-\beta H_N(\sigma)}}{Z_{N,\beta}} \qquad \text{with} \qquad Z_{N,\beta} = \sum_{\sigma \in \mathcal{S}_N} e^{-\beta H_N(\sigma)}$$

were $\beta \in (0, \infty)$ is the inverse temperature and $Z_{N,\beta}$ the partition function.
Discrete time Glauber dynamics on $\mathcal{S}_N$ with Metropolis transition probabilities

$$p_N(\sigma, \sigma') = \begin{cases} \frac{1}{N} \exp(-\beta [H_N(\sigma') - H_N(\sigma)]_+) & \text{if } \sigma \sim \sigma', \\ 1 - \sum_{\eta \neq \sigma} p(\sigma, \eta) & \text{if } \sigma = \sigma', \\ 0 & \text{otherwise.} \end{cases}$$



$\mu_{N,\beta}$ is the unique invariant and reversible measure.

Magnetization in the Curie-Weiss model

The fact that this is a mean-field model is expressed by the fact that $H_N(\sigma)$ depends on σ only through the empirical magnetization

$$m_N(\sigma) = \frac{1}{N}\sum_{i\in[N]}\sigma_i, \qquad \mathcal{S}_N[m] := m_N^{-1}(m).$$

m_N takes values in $\Gamma_N = \left\{-1, -1+\frac{2}{N}, ..., 1-\frac{2}{N}, 1\right\}$. Hence

$$H_N^{\mathsf{CW}}(\sigma) = -N\left(\tfrac{1}{2}m_N(\sigma)^2 + h\, m_N(\sigma)\right) =: NE(m_N(\sigma)).$$

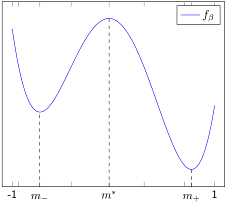
Mesoscopic measure on Γ_N:

$$\mathcal{Q}_{N,\beta}^{\mathsf{CW}}(m) = \mu_{N,\beta}^{\mathsf{CW}} \circ m_N^{-1}(m) = \frac{\mathrm{e}^{-\beta N f_{N,\beta}(m)}}{Z_{N,\beta}^{\mathsf{CW}}}$$

where $f_{N,\beta}$ is the free energy and I_N is the entropy

$$f_{N,\beta}(m) = E(m) + \beta^{-1} I_N(m)$$

Metastability for the Curie–Weiss model



- $\lim_{N\to\infty} f_{N,\beta}(m) = f_\beta(m)$
- Hitting time of A

$$\tau_A = \inf\{t > 0 : \sigma_t \in A\}.$$

- $(m_-(N), m^*(N), m_+(N))$ are the closest points in Γ_N to (m_-, m^*, m_+).
- $\mathbb{E}^{\mathsf{CW}}_{m_-(N)}$ is the expectation w.r.t. the Markov process for the CW model with Glauber dynamics starting in $m_-(N)$.

Theorem (Mean metastable exit time)

For $\beta > 1$ and $h > 0$ small enough, as $N \to \infty$,

$$\mathbb{E}^{CW}_{m_-(N)}[\tau_{m_+(N)}] = \exp\Big(\beta N\left[f_\beta(m^*) - f_\beta(m_-)\right]\Big) \times \frac{\pi}{1-m^*}\sqrt{\frac{1-{m^*}^2}{1-m_-^2}}\,\frac{N(1+o(1))}{\beta\sqrt{f''_\beta(m_-)\left(-f''_\beta(m^*)\right)}}$$

Results: main theorem

Last exit-biased distribution

$$\nu_{A,B}(\sigma) = \frac{\mu_N(\sigma)\mathbb{P}_\sigma(\tau_B < \tau_A)}{\sum_{\sigma\in A}\mu_N(\sigma)\mathbb{P}_\sigma(\tau_B < \tau_A)}, \qquad \sigma \in A$$

Notation: $\nu_{m_-,m_+} = \nu_{\mathcal{S}_N[m_-(N)],\mathcal{S}_N[m_+(N)]}$

$\mathbb{P}_J$ is the law of the random couplings (or the law of the ER random graph).

Theorem (Metastable exit time for the RDCW model)

For $\beta > 1$, $h > 0$ small enough and for $s > 0$, there exist absolute constants $k_1, k_2 > 0$ and $C_1(p,\beta) < C_2(p,\beta,h)$ independent of N, such that

$$\lim_{N\uparrow\infty} \mathbb{P}_J\left(C_1 \mathrm{e}^{-s} \leq \frac{\mathbb{E}_{\nu_{m_-,m_+}}\left[\tau_{\mathcal{S}_N[m_+(N)]}\right]}{\mathbb{E}^{CW}_{m_-(N)}\left[\tau_{m_+(N)}\right]} \leq C_2 \mathrm{e}^{s}\right) \geq 1 - k_1 \mathrm{e}^{-k_2 s^2}.$$

[*A. Bovier, S. Marello, and E. P., "Metastability for the dilute Curie–Weiss model with Glauber dynamics", preprint 2019, arXiv: 1912.10699*]

Background

Equilibrium RDCW model:

- Bovier and Gayrard, '93: prove that the RDCW free energy converges to that of the CW model (in the thermodynamic limit), when p decreases with the system size in a certain way.

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Metastability for interacting particle systems on random graphs:

- Dommers, den Hollander, Jovanovski, and Nardi, '17: random regular graph and configuration model with Glauber dynamics, in the limit as $\beta \to \infty$ and the number of vertices is fixed.
- den Hollander and Jovanovski, '19: Erdős–Rényi random graph for fixed temperature in the thermodynamic limit. It is exactly the RDCW model.

Results: discussion

Theorem (Metastable exit time for the RDCW model)

For $\beta > 1$, $h > 0$ small enough and for $s > 0$, there exist absolute constants $k_1, k_2 > 0$ and $C_1(p, \beta) < C_2(p, \beta, h)$ independent of N, such that

$$\lim_{N \uparrow \infty} \mathbb{P}_J \left(C_1 \mathrm{e}^{-s} \leq \frac{\mathbb{E}_{\nu_{m_-, m_+}} \left[\tau_{\mathcal{S}_N[m_+(N)]} \right]}{\mathbb{E}^{CW}_{m_-(N)} \left[\tau_{m_+(N)} \right]} \leq C_2 \mathrm{e}^{s} \right) \geq 1 - k_1 \mathrm{e}^{-k_2 s^2}.$$

Comparison with den Hollander and Jovanovski:
With $\mathbb{P}_J \to 1$ as $N \to \infty$, uniformly in $\xi \in \mathcal{S}_N[m_-(N)]$,

$$\mathbb{E}_\xi \left[\tau_{\mathcal{S}_N[m_+(N)]} \right] = N^{\mathcal{E}_N} \exp \Big(\beta N \left[f_\beta(m^*) - f_\beta(m_-) \right] \Big),$$

i.e. they prove that the multiplicative error term is at most *polynomial* in N. They do not know how to identify the *random* prefactor. They use pathwise approach to metastability.

Mesoscopic measure and closeness to CW

Obtain a mesoscopic description in terms of the magnetization

$$m_N(\sigma) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i \qquad \text{for } \sigma \in \mathcal{S}_N$$

$$\mathcal{Q}_{N,\beta}(m) = \mu_{N,\beta} \circ m_N^{-1}(m) = \mu_{N,\beta}(\mathcal{S}_N[m]) \qquad \text{for } m \in \Gamma_N$$

Proposition

For every $m \in \Gamma_N$, asymptotically for $N \to \infty$,

$$Z_N \mathcal{Q}_N(m) \leq \mathrm{e}^{\alpha}\, Z_N^{\mathsf{CW}}\, \mathcal{Q}_N^{\mathsf{CW}}(m)\, \exp\left(\mathcal{Y}_{N,m}\right)(1 + o(1)),$$

where $\mathcal{Y}_{N,m}$ is a sub-Gaussian random variable, i.e. for any $\beta > 0$, any $s > 0$,

$$\mathbb{P}_J\Big(|\mathcal{Y}_{N,m}| \geq s\Big) \leq c_1 \exp\Big(-2c_2\frac{p^2}{\beta^2}s^2\Big).$$

Same lower bound with κ instead of α.

Mesoscopic measure and closeness to CW

Target result

$$Z_N \mathcal{Q}_N(m) \approx c\, Z_N^{\mathsf{CW}}\, \mathcal{Q}_N^{\mathsf{CW}}(m)\, \exp\left(\mathcal{Y}_{N,m}\right)(1+o(1))$$

($\approx$ means we have upper bound with e^{α} and lower bound with e^{κ})

$$\begin{aligned} Z_N \mathcal{Q}_N(m) &= \sum_{\sigma\in\mathcal{S}_N[m]} \mathrm{e}^{-\beta H_N(\sigma)} = \mathrm{e}^{-\beta N E(m)} \sum_{\sigma\in\mathcal{S}_N[m]} \mathrm{e}^{-\beta[H_N(\sigma)-H_N^{\mathsf{CW}}(\sigma)]} \\ &=: \mathrm{e}^{-\beta N E(m)} \cdot \exp\left(N F_{N,m}\right) \\ &= \mathrm{e}^{-\beta N E(m)} \cdot \exp\left(\mathbb{E}(N F_{N,m})\right) \exp\left(N\left[F_{N,m} - \mathbb{E}F_{N,m}\right]\right) \end{aligned}$$

Recall:

$$Z_N^{\mathsf{CW}} \mathcal{Q}_N^{\mathsf{CW}}(m) = \mathrm{e}^{-\beta N f_N(m)} = \mathrm{e}^{-\beta N E(m)} \cdot |\mathcal{S}_N[m]|$$

Mesoscopic measure and closeness to CW

Target result

$$Z_N \mathcal{Q}_N(m) \approx c \boxed{Z_N^{\mathsf{CW}} \mathcal{Q}_N^{\mathsf{CW}}(m)} \exp\left(\mathcal{Y}_{N,m}\right) (1 + o(1))$$

$$\begin{aligned} Z_N \mathcal{Q}_N(m) &= \sum_{\sigma \in \mathcal{S}_N[m]} \mathrm{e}^{-\beta H_N(\sigma)} = \mathrm{e}^{-\beta N E(m)} \sum_{\sigma \in \mathcal{S}_N[m]} \mathrm{e}^{-\beta [H_N(\sigma) - H_N^{\mathsf{CW}}(\sigma)]} \\ &=: \mathrm{e}^{-\beta N E(m)} \cdot \exp\left(N F_{N,m}\right) \\ &= \boxed{\mathrm{e}^{-\beta N E(m)} \cdot \exp\left(\mathbb{E}(N F_{N,m})\right)} \exp\left(N\left[F_{N,m} - \mathbb{E} F_{N,m}\right]\right) \end{aligned}$$

Recall:

$$Z_N^{\mathsf{CW}} \mathcal{Q}_N^{\mathsf{CW}}(m) = \mathrm{e}^{-\beta N f_N(m)} = \boxed{\mathrm{e}^{-\beta N E(m)} \cdot |\mathcal{S}_N[m]|}$$

Mesoscopic measure and closeness to CW

Sub-Gaussian bounds on the stochastic part.

Proposition

$N\left[F_{N,m} - \mathbb{E}F_{N,m}\right]$ is sub-Gaussian, i.e. for any β, $s > 0$

$$\mathbb{P}_J\Big(|N(F_{N,m} - \mathbb{E}F_{N,m})| \geq s\Big) \leq c_1 \exp\Big(-2c_2\frac{p^2}{\beta^2}s^2\Big).$$

Proof: use the following result

Theorem (Talagrand's concentration inequality)

Let $G : \mathbb{R}^n \to \mathbb{R}$ be a 1-Lipschitz and convex function and $g = (g_i)_{i\in[n]}$ be independent r.v., uniformly bounded by $K > 0$. Then, for any $t \geq 0$,

$$\mathbb{P}\Big(|G(g) - \mathbb{E}G(g)| \geq tK\Big) \leq c_1 \exp\big(-c_2t^2\big).$$

Apply the theorem to the free energies $F_{N,m}$ as a function of the coupling constants $(J_{ij} - p)_{ij}$ and use $G = \frac{p\sqrt{2}}{\beta}NF_{N,m}$.

Mesoscopic measure and closeness to CW

Asymptotic bounds on the deterministic part.

Proposition

$$\mathrm{e}^{\kappa}|\mathcal{S}_N[m]|(1+o(1)) \le \exp(\mathbb{E}[N\,F_{N,m}]) \le \mathrm{e}^{\alpha}|\mathcal{S}_N[m]|(1+o(1))$$

$$\exp(NF_{N,m}) = \sum_{\sigma\in\mathcal{S}_N[m]} \exp\left[-\frac{\beta}{Np}\sum_{1\le i<j\le N}(J_{ij}-p)\sigma_i\sigma_j\right]$$

$$\begin{aligned}\mathbb{E}[\exp(x(J_{ij}-p))] &= 1 + x\mathbb{E}(J_{ij}-p) + \frac{x^2}{2}\mathbb{E}(J_{ij}-p)^2 + o_0(x^2)\\ &= 1 + \frac{x^2}{2}p(1-p) + o_0(x^2)\end{aligned}$$

Upper bound:

- $\mathbb{E}[\exp(N\,F_{N,m})]$
- Jensen's inequality

Mesoscopic measure and closeness to CW

Asymptotic bounds on the deterministic part.

Proposition

$$e^{\kappa}|\mathcal{S}_N[m]|(1+o(1)) \leq \exp(\mathbb{E}[N\,F_{N,m}]) \leq e^{\alpha}|\mathcal{S}_N[m]|(1+o(1))$$

$$\exp(NF_{N,m}) = \sum_{\sigma\in\mathcal{S}_N[m]} \exp\left[-\frac{\beta}{Np}\sum_{1\leq i<j\leq N}(J_{ij}-p)\sigma_i\sigma_j\right]$$

Lower bound:

- $\mathbb{E}[\exp(2NF_{N,m})] \leq e^{2\alpha}\mathbb{E}^2[\exp(NF_{N,m})]$
- Paley–Zygmund inequality, $\eta \in (0,1)$

$$\mathbb{P}\big(X \geq \eta\,\mathbb{E}X\big) \geq (1-\eta)^2\frac{(\mathbb{E}X)^2}{\mathbb{E}X^2},$$

- Talagrand's concentration inequality

Potential theoretic approach (Bovier, Eckhoff, Gayrard and Klein, 2001)

Translates the problem of understanding the metastable behaviour of Markov processes to the study of capacities of electric networks. Link between **mean metastable crossover time** and **capacity**.
For A, B disjoint subsets of $\mathcal{S}_N$, the key formula is

$$\mathbb{E}_{\nu_{A,B}}[\tau_B] = \sum_{\sigma \in A} \nu_{A,B}(\sigma)\mathbb{E}_\sigma[\tau_B] = \frac{1}{\mathrm{cap}(A,B)} \sum_{\sigma' \in \mathcal{S}_N} \mu_N(\sigma')h_{AB}(\sigma'),$$

where

$$\mathrm{cap}(A,B) = \sum_{\sigma \in A} \mu_N(\sigma)\mathbb{P}_\sigma(\tau_B < \tau_A)$$

and h_{AB} is called *harmonic function*

$$h_{AB}(\sigma) = \begin{cases} \mathbb{P}_\sigma(\tau_A < \tau_B) & \sigma \in \mathcal{S}_N \setminus (A \cup B), \\ \mathbb{1}_A(\sigma) & \sigma \in A \cup B. \end{cases}$$

Capacity estimates

We are interested in $$\mathbb{E}_{\nu_{A,B}}[\tau_B] = \frac{1}{\boxed{\mathrm{cap}(A,B)}} \sum_{\sigma' \in \mathcal{S}_N} \mu_N(\sigma') h_{AB}(\sigma')$$

with $A = \mathcal{S}_N[m_-(N)], B = \mathcal{S}_N[m_+(N)]$

Dirichlet principle

$$\mathrm{cap}(A,B) = \inf_{\substack{g:\mathcal{S}_N \to [0,1] \\ g|_A = 1, g|_B = 0}} \frac{1}{2} \sum_{\sigma,\sigma' \in \mathcal{S}_N} \mu_N(\sigma) p_N(\sigma,\sigma')[g(\sigma) - g(\sigma')]^2.$$

Thomson principle

$$\mathrm{cap}(A,B) = \sup_{\phi \in \mathcal{U}_{AB}} \frac{1}{\mathcal{D}(\phi)}, \quad \mathcal{D}(\phi) = \sum_{(\sigma,\sigma') \in E} \frac{\phi(\sigma,\sigma')^2}{\mu_N(\sigma) p_N(\sigma,\sigma')}$$

Idea

Estimate capacity in terms of the capacity of the CW model

Thank you for your attention!