PHASE TRANSITIONS AND FINITARY CODINGS

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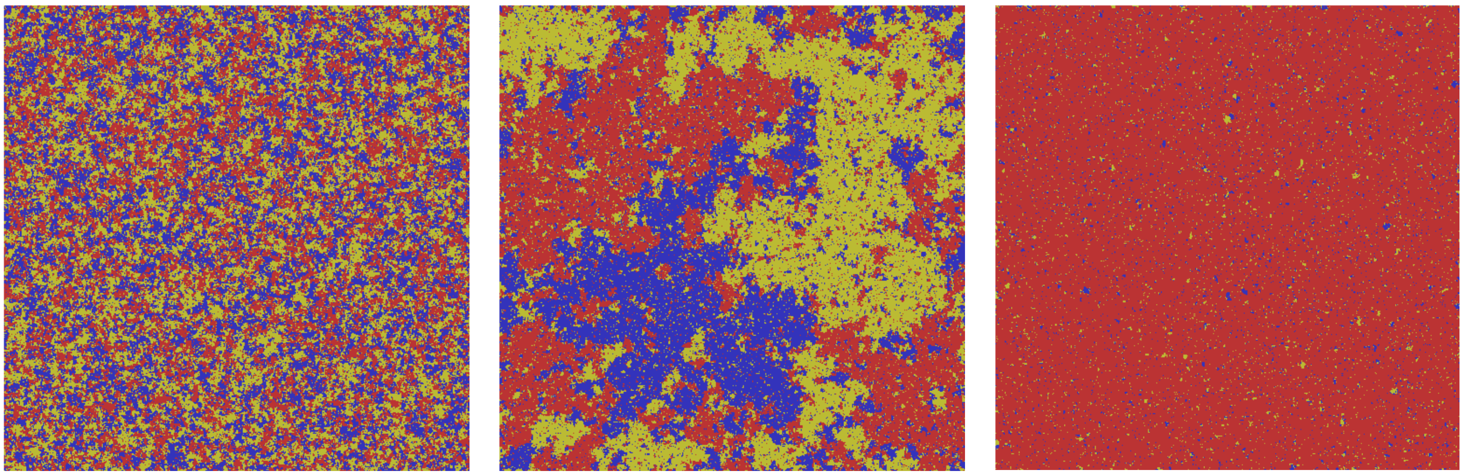
Phase transitions



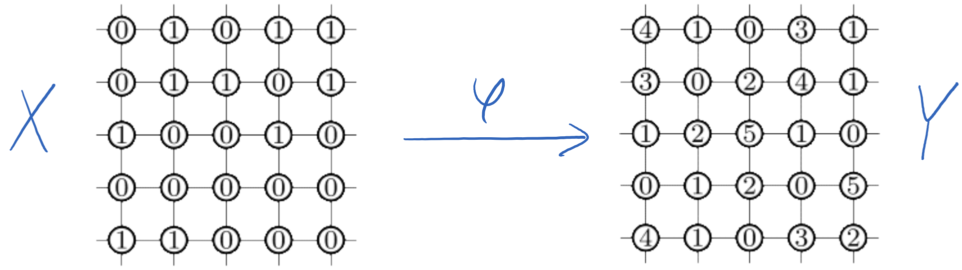
Finitary codings

- Ergodic theory — Akcoglu, Denker, Fiebig, del Junco, Kalikow, Keane, Krieger, Rahe, Rudolph, Schmidt, Serafin, Smorodinsky, Weiss
- Probability — Angel, Benjamini, Gurel-Gurevich, Harvey, den Hollander, Holroyd, Hutchcroft, Keane, Kosloff, Levy, Lyons, Meyerovitch, Nazarov, Peled, Peres, Romik, Schramm, Soo, Steif, Timar, Wilson
- Statistical mechanics — van den Berg, Häggström, Harel, Ray, Steif

Closely related to
- Phase transitions
- Algorithms for perfect sampling (exact simulation)

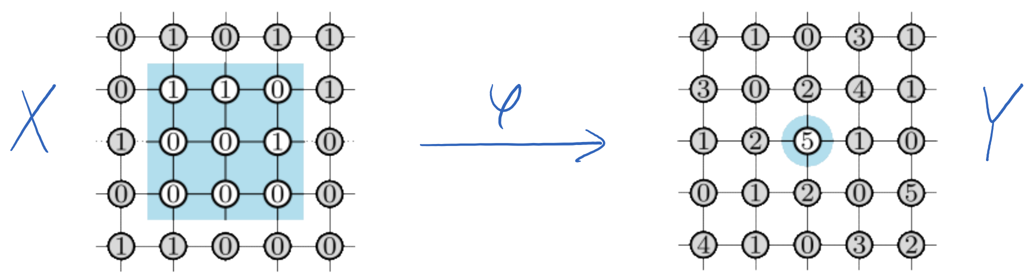
Introduction:

- $X = (X_v)_{v \in \mathbb{Z}^d}$, $Y = (Y_v)_{v \in \mathbb{Z}^d}$ (translation invariant) [e.g. X IID, Y trans-inv. Gibbs measure]



- Y is a **factor** of X if: $Y = \varphi(X)$ for some φ s.t.
  - (*) φ is measurable
  - (*) φ is equivariant
    [commutes with translations $\varphi \circ T = T \circ \varphi \quad \forall T$ trans.]

- φ is **finitary** if: Y_0 can be determined from the values of X on a finite box Λ_R (which may depend on the input X)
  [i.e. for ν-a.e. x there exists $R < \infty$ s.t. $\varphi(x)_0$ only depends on $(x_u)_{u \in \Lambda_R}$]



- **Example**: X IID Bernoulli$(1/2)$
  - $Y_v := \min \{ \|u-v\|_\infty : X_u = X_v ,\ u \neq v \}$ — finitary factor
  - $Y_v := \max \{ k \geq 0 : X_u = X_v \text{ for } \|u-v\|_\infty = k \}$ — factor, non-finitary

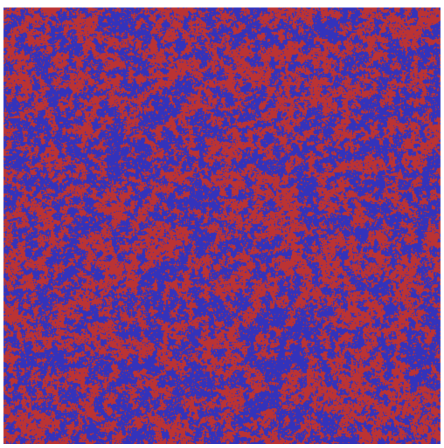
- **Question**: Given Y, can Y be represented as

• Question: Given μ, can μ be represented as

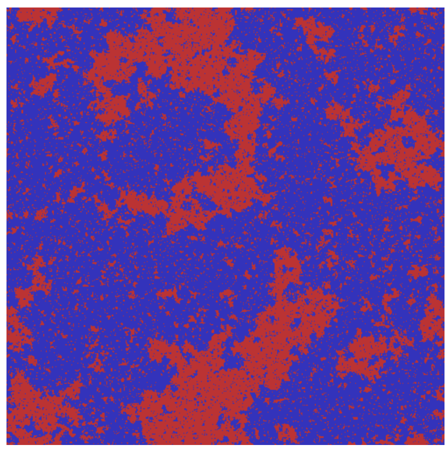
- a factor of IID? (Bernoulli)
- a finitary factor of IID? (FFIID)

(and if so, is there an efficient coding?)

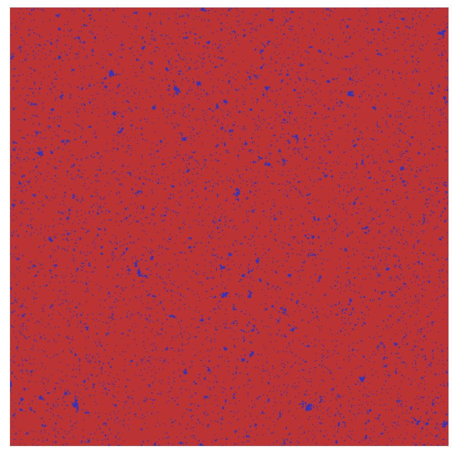
Ferromagnetic Ising model on $\mathbb{Z}^d$: $(d \geq 2)$



$\beta < \beta_c$



$\beta = \beta_c$



$\beta > \beta_c$

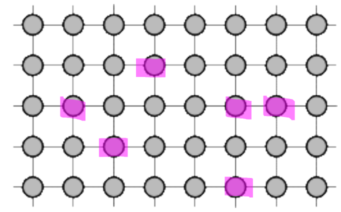
- μ^+ is a factor of IID for all β. (Ornstein, Weiss ~1977)

  $\Rightarrow$ factor of IID cannot "detect" the phase transition.

- μ^+ is FFIID $\iff$ $\mu^+ = \mu^-$ (van den Berg, Steif 1999)

  $[\iff \beta \leq \beta_c]$

What is the algorithm that produces the finitary coding?



- Discrete-time Glauber dynamics

  $\Rightarrow$ Starting from any τ, we get $\omega_1^\tau, \omega_2^\tau, \omega_3^\tau, \ldots$

- Coupling from the past (CFTP) (Propp-Wilson '96)

  Instead of starting at time 0 and running to time n, start at time $-n$ and run to time 0.

  $$\omega_n^\tau = \omega_{0\to n}^\tau \qquad \Rightarrow \qquad \hat{\omega}_n^\tau = \omega_{-n\to 0}^\tau .$$

- Monotonicity

  $\Rightarrow \quad \hat{\omega}_1^+ \geq \hat{\omega}_2^+ \geq \hat{\omega}_3^+ \geq \ldots$

  $\Rightarrow \quad \hat{\omega}^+ := \lim_{n\to\infty} \hat{\omega}_n^+$ exists almost surely

- $\hat{\omega}^+ \sim \mu^+ , \quad \hat{\omega}^- \sim \mu^- , \quad \hat{\omega}^- \leq \hat{\omega}^+$ a.s.

- $\mu^+ = \mu^- \Rightarrow \hat{\omega}^- = \hat{\omega}^+$ a.s.

  $\Rightarrow$ $T := \min\{n \geq 1 : \hat{\omega}_n^-(0) = \hat{\omega}_n^+(0)\}$ is a.s. finite.

van den Berg and Steiff

- $\beta = \beta_c$: any finitary coding has infinite expected coding volume : $\mathbb{E}R^d = \infty$. (+ Peres)
- $\beta < \beta_c$: can use finite-valued IID OR can get exponential tails for coding radius.

  [uses that T has exp. tails (Martinelli-Olivieri 94)]

each answers

- $\beta < \beta_c$: can use finite-valued IID AND (… 2018) each answers a question of van den Berg and Steif

- $\beta < \beta_c$: can use finite-valued IID AND can get stretched-exp. tails for coding radius. (S. 2018)
- $\beta = \beta_c$: can use finite-valued IID. (Meyerovitch, S. 2020+)

[More generally, if Y is FFIID (and Y is finite-valued) then can use finite-valued IID whose entropy is $h(Y)+\varepsilon$.]

- $\beta > \beta_c$: "gradient" of μ^+ is FFIID (Ray, S. 2019)

[implies that "energy" in a box has volume-order large deviation estimate]

Other models: Potts, colorings, hard-core, random-cluster, ...

Meta-question (the van den Berg - Steif phenomenon):

When does FFIID $\longleftrightarrow$ phase transition ?

Question (van den Berg and Steif 99): If a Markov random field is "unique", must it be a factor of IID?

Question (Steif): If a subshift of finite type has a unique measure of maximal entropy which is a factor of IID, must it also be FFIID ?

taken from "Open problems in symbolic dynamics" Boyle 2008

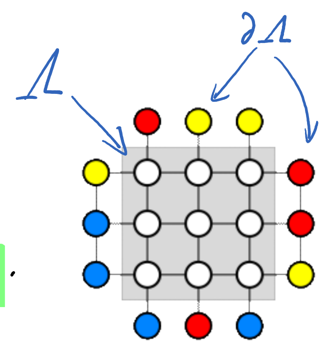
Ferromagnetic Ising model

Nearest-neighbor interactions,
Markov random field (MRF)
[e.g. Potts, colorings]

Monotonicity, FKG
[e.g. random-cluster model]

Markov random fields:

- $X = (X_v)_{v \in \mathbb{Z}^d}$ is a MRF if given X_{Λ^c}, the conditional law of X_Λ only depends on $X_{\partial\Lambda}$.



- Suppose X is a translation-invariant MRF taking values in S.

- Theorem (van den Berg, Steif 99, S. 2018)

  Non-uniqueness + mild technical assumption [e.g. finite energy] $\Rightarrow$ X is not FFIID

- High noise:

$$\gamma := \sum_{s \in S} \min_{\tau \in S^{\partial v}} \mathbb{P}(X_v = s \mid X_{\partial v} = \tau)$$

  [called the multigamma admissibility]

- Theorem (Häggström, Steif 2000)

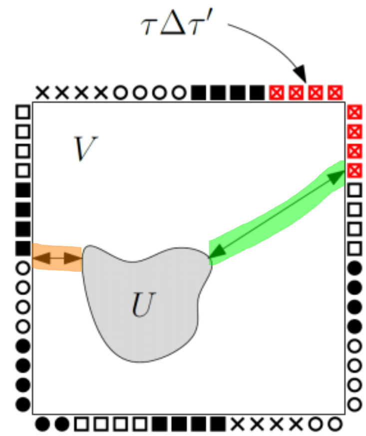
$\gamma > 1 - \frac{1}{2d} \implies X$ is FFIID with exp. tails

$\underbrace{\gamma > 1 - \frac{1}{2d}}_{\text{stronger than Dobrushin uniqueness assumption}}$

[e.g. the unique Potts measure when $|\beta| < \beta_0(q,d) \approx \frac{c(q)}{d^2}$]

- Proper q-colorings : theorem does not apply since $\gamma = 0$,
  but same result holds when $q \geq 4d(d+1)$.
  (Huber 99, S. 2018)

- Weak and strong spatial mixing



WSM : $\|P^{\tau}_{V,U} - P^{\tau'}_{V,U}\|_{TV} \leq C|U|\, e^{-c\, \mathrm{dist}(U, \partial V)}$

SSM : $\|P^{\tau}_{V,U} - P^{\tau'}_{V,U}\|_{TV} \leq C|U|\, e^{-c\, \mathrm{dist}(U, \tau \Delta \tau')}$

[Dobrushin uniqueness $\implies$ SSM]

- Theorem (S. 2018)
  - WSM $\implies$ X is FFIID with power-law tails
  - SSM $\implies$ X is FFIID with exp. tails

  [e.g. ferromagnetic Potts for all $\beta < \beta_c$
  e.g. proper q-colorings when $q > 3.53d$]

  - WSM can be relaxed to $\exists\, \delta \in (0,1)\ \forall n\ \gamma_{\Lambda_n, \Lambda_{\delta n}} \geq c > 0$.

where $\gamma_{v,u} = \sum_{w \in S^u} \min_{\tau \in S^{\partial v}} \mathbb{P}(X_u = w \mid X_{\partial v} = \tau)$

[e.g. Potts $d=2$, $\beta = \beta_c$ is FFIID $\iff$ $q \le 4$]

THANK YOU !!!