Intro to Counting and Sampling

Eric Vigoda

Georgia Tech

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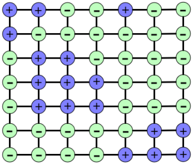
Talk Outline

1. Setting
2. Path Coupling and Dobrushin Uniqueness:
   Rapid Mixing for very high temperature.
3. Strong Spatial Mixing (SSM):
   Ferro Ising: $O(n \log n)$ mixing on boxes of $\mathbb{Z}^2$
4. Mossel-Sly for General Graphs:
   Ferro Ising: $O(n \log n)$ mixing for general graphs
5. Correlation Decay:
   2-spin antiferro: FPTAS for general graphs
6. Spectral Independence:
   2-spin antiferro: $O(n \log n)$ mixing for general graphs

Ising Model

Consider graph $G = (V, E)$ as $L \times L$ box of $\mathbb{Z}^2$, $n = |V|$:



Configurations: $\Omega = \{-1, +1\}^V$.

Inverse temperature β. For $\sigma \in \Omega$:

Monochromatic edges: $M(\sigma) = |\{(v, w) \in E : \sigma(v) = \sigma(w)\}|$

Sampling: Gibbs distribution: $\mu(\sigma) = \frac{\exp(\beta M(\sigma))}{Z}$

Counting: Partition function: $Z = Z_G = \sum_{\sigma \in \Omega} \exp(\beta M(\sigma))$.

$\beta > 0$ is ferromagnetic and $\beta < 0$ is anti-ferromagnetic

Markov Chain for Ising Model

Glauber Dynamics: For $G = (V, E)$, MC (X_t) on $\Omega = \{-1, +1\}^V$.

From $X_t \in \Omega$:

- Choose $v \in V$ uniformly at random.
- For all $w \neq v$, set $X_{t+1}(w) = X_t(w)$.
- Choose $X_{t+1}(v)$ from marginal conditional on neighbors spin:

$$\mu(\sigma(v) | \sigma(w) = X_{t+1}(w),\ w \in N(v)).$$

Markov Chain for Ising Model

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Stationary distribution π is Gibbs distribution μ.

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How fast does it converge to π?

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Stationary distribution π is Gibbs distribution μ.

How fast does it converge to π?

$$T_{\mathrm{mix}}(\epsilon) = \max_{X_0 \in \Omega} \min\{t : d_{\mathrm{TV}}(P^t(X_0, \cdot), \pi) \leq \epsilon\}.$$

For dist. μ, ν on Ω, $d_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)| = \max_{S \subset \Omega} \mu(S) - \nu(S)$.

Mixing time: $T_{\mathrm{mix}} := T_{\mathrm{mix}}(1/4)$

Sub-multiplicative: $T_{\mathrm{mix}}(\epsilon) \leq \lceil \log_2(1/\epsilon) \rceil T_{\mathrm{mix}}$

Counting $\leftrightarrow$ Sampling

Let $\mathcal{G}_\Delta$ denote all graphs of maximum degree Δ.

Approx sampler μ_G, $\forall G \in \mathcal{G}_\Delta \leftrightarrow$ Approx counting Z_G $\forall G \in \mathcal{G}_\Delta$.

Approximate sampler:

Given graph $G = (V, E) \in \mathcal{G}_\Delta$ and $\delta > 0$, outputs X where

$$d_{\mathsf{TV}}(X, \mu_G) \leq \delta,$$

in time poly$(|V|, \log(1/\delta))$.

FPRAS for approximate counting:

Given graph $G = (V, E)$ of maximum degree Δ and $\delta, \epsilon > 0$, outputs OUT where

$$\mathbf{Pr}\left((1-\epsilon)OUT \leq Z_G \leq (1+\epsilon)OUT\right) \geq 1-\delta,$$

in time poly$(|V|, 1/\epsilon, \log(1/\delta))$.

FPTAS $=$ FPRAS with $\delta = 0$.

Approx counting via Sampling

Simulated annealing: Let $\beta_0 = \beta > \beta_1 > \cdots > \beta_{\ell-1} > \beta_\ell \approx \infty$.
Simple scheme: $\beta_i = \beta_{i-1}(1 + 1/n)$.

$$Z_G(\beta) = \frac{Z_G(\beta_0)}{Z_G(\beta_1)} \frac{Z_G(\beta_1)}{Z_G(\beta_2)} \cdots \frac{Z_G(\beta_{\ell-1})}{Z_G(\beta_\ell)} 2^n.$$

Estimate $\frac{Z_G(\beta_i)}{Z_G(\beta_{i-1})}$ by sampling $\mu(\beta_{i-1})$, outputting $X_i = \frac{w_{\beta_i}(\sigma)}{w_{\beta_{i-1}}(\sigma)}$.
If $Var(X_i) = O(1)$ for all i, then, $O((\ell/\epsilon)^2)$ total samples suffices.

Better scheme: exists $\ell = O(\sqrt{n} \times \text{poly}(\log n))$.

[Stefankovic-Vempala-V '09], [Huber '15],[Kolmogorov '18]

$T_{\text{mix}} = O(n \log n) \implies$ FPRAS in $O((n/\epsilon)^2 \log n)$ time.

Path Coupling/Dobrushin Uniqueness

Bounding mixing time of Glauber dynamics.

Simple/classical technique:

Path coupling and Dobrushin uniqueness condition

How well do these approaches perform?

For now: Ferromagnetic Ising model.

Glauber dynamics for Ising Model

For $G = (V, E)$, let $\Omega = \{-1, +1\}^V$.

From $X_t \in \Omega$:

- Choose $v \in V$ uniformly at random.
- For all $w \neq v$, set $X_{t+1}(w) = X_t(w)$.
- Choose $X_{t+1}(v)$ from marginal conditional on neighbors spin.

$$T_{\mathrm{mix}} = \max_{X_0 \in \Omega} \min\{t : d_{\mathsf{TV}}(P^t(X_0, \cdot), \pi) \leq 1/4\}.$$

Coupling for bounding T_{mix}

For all X_t, Y_t, define a coupling: $(X_t, Y_t) \to (X_{t+1}, Y_{t+1})$.

Look at Hamming distance: $H_t = |\{v \in V : X_t(v) \neq Y_t(v)\}|$.

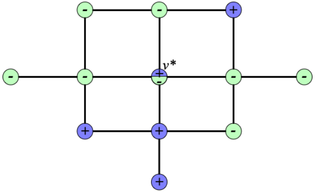
$$\text{If for all } X_t, Y_t \in \Omega, \quad \mathbb{E}\left[H_{t+1} | X_t, Y_t\right] \leq (1 - 1/n)H_t,$$
$$\text{then } T_{\text{mix}} = O(n \log n).$$

$d_{\text{TV}}(X_T, Y_T) \leq \mathbf{Pr}\left(X_T \neq Y_T\right) \leq \mathbb{E}\left[H_t\right] \leq H_0(1 - 1/n)^T \leq n \exp(-T/n) \leq 1/4.$

Path coupling [Bubley-Dyer '97]: Suffices to consider pairs where $H_t = 1$.
Idea: Couplings compose and linearity of expectation.

Path Coupling on $\mathbb{Z}^2$:

Consider a pair (X_t, Y_t) that differ at exactly one vertex v^*:



Update v^* then $H(X_{t+1}, Y_{t+1}) = 0$.
For $w \in N(v^*)$. Let d_w^+ (d_w^-) be number of $+$ (and $-$) neighbors in Y_t.

Update $w \in N(v^*)$ then $H(X_{t+1}, Y_{t+1}) = 2$ with probability:

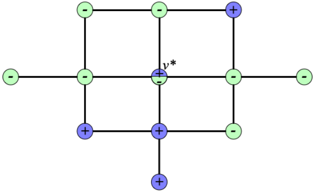
$$\alpha(w) := \frac{\exp(\beta(d_w^+ + 1))}{\exp(\beta(d_w^+ + 1)) + \exp(\beta(d_w^- - 1))} - \frac{\exp(\beta d_w^+)}{\exp(\beta d_w^+) + \exp(\beta(d_w^-))}$$

$$\mathbb{E}\left[H(X_{t+1}, Y_{t+1})\right] \leq 1 - \frac{1}{n} + \frac{1}{n} \sum_{w \in N(v)} \alpha(w).$$

Worst case $d^+ = d^-$. When $d = 4$ works for $\beta < .55$.

PATH COUPLING ON $\mathbb{Z}^2$:

Consider a pair (X_t, Y_t) that differ at exactly one vertex v^*:



Update v^* then $H(X_{t+1}, Y_{t+1}) = 0$.
For $w \in N(v^*)$. Let d_w^+ (d_w^-) be number of $+$ (and $-$) neighbors in Y_t.

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Worst case $d^+ = d^-$. When $d = 4$ works for $\beta < .55$.
Goal: All $\beta < \beta_c := \ln(1 + \sqrt{2})$.

Path Coupling vs. Dobrushin Uniqueness

For a configuration $\sigma \in \{+,-\}^V$ and $w \in V$, let

$$\sigma^w(z) = \begin{cases} \sigma(z) & \text{for } z \neq w \\ -\sigma(w) & \text{for } z = w. \end{cases}$$

What's the effect of disagreement at v on neighbors of v?

Path coupling condition:

$$\max_w \max_{\sigma,\sigma^w} \sum_{z \in N(w)} d_{\mathrm{TV}}\left[\mu\Big(\sigma(z)|\sigma(N(z))\Big), \mu\Big(\sigma(z)|\sigma^w(N(z))\Big)\right] < 1.$$

Dobrushin uniqueness:

$$\max_w \sum_{z \in N(w)} \max_{\sigma,\sigma^w} d_{\mathrm{TV}}\left[\mu\Big(\sigma(z)|\sigma(N(z))\Big), \mu\Big(\sigma(z)|\sigma^w(N(z))\Big)\right] < 1.$$

Can we prove rapid mixing for all $\beta < \beta_c(\mathbb{Z}^2)$?

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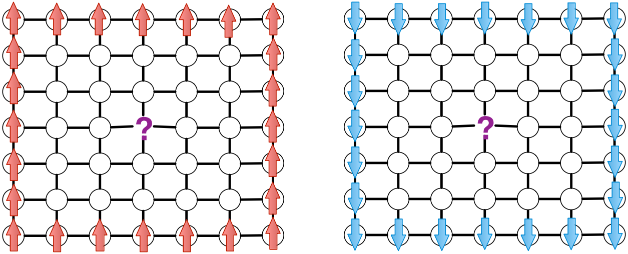
Uniqueness Phase Transition?

Influence of boundary:

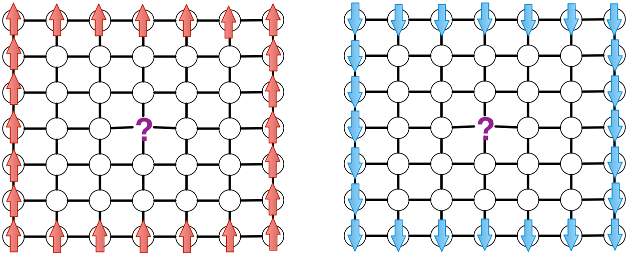


Let $p_L^+ = \mathbf{Pr}(\text{Origin has } + \mid \text{all } + \text{ boundary for } L \times L \text{ box})$.

Let $p_L^- = \mathbf{Pr}(\text{Origin has } + \mid \text{all - boundary for } L \times L \text{ box})$.

Uniqueness Phase Transition?

Influence of boundary:



For ferromagnetic Ising model, critical point $\beta_c(\mathbb{Z}^2) = \ln(1+\sqrt{2})$:

For all $\beta < \beta_c(\mathbb{Z}^2)$, $\lim_{L\to\infty} \mathrm{p}_L^+ - \mathrm{p}_L^- = 0$ uniqueness

For all $\beta > \beta_c(\mathbb{Z}^2)$, $\lim_{L\to\infty} \mathrm{p}_L^+ - \mathrm{p}_L^- > 0$ non-uniqueness

Ferro Potts: $\beta_c(\mathbb{Z}^2) = \ln(1+\sqrt{q})$ [Beffara,Duminil-Copin '12]

Phase Transition on Regular Tree

For Δ-regular tree of height ℓ:

$$p_\ell^+ = \mathbf{Pr}(\text{ root has spin } + \mid \text{ leaves have spin } +)$$
$$p_\ell^- = \mathbf{Pr}(\text{ root has spin } + \mid \text{ leaves have spin } -)$$

Does $\lim_{\ell\to\infty} p_\ell^+ = \lim_{\ell\to\infty} p_\ell^-$?

- Uniqueness ($\beta \le \beta_c(\mathbb{T}_\Delta)$): No boundary affects root.
- Non-Uniqueness ($\beta > \beta_c(\mathbb{T}_\Delta)$): Exist boundaries affect root.

[Häggström '96]: $\beta_c(\mathbb{T}_\Delta) = \ln\left(\frac{\Delta}{\Delta-2}\right)$

Glauber dynamics on $\mathbb{Z}^2$

For $L \times L$ box of $\mathbb{Z}^2$ with volume $n = |V|$:

high temp. $\beta < \beta_c$ — β_c — low temp. $\beta > \beta_c$

$O(n \log n)$ mixing of Glauber for all b.c.

$\exp(\Omega(\sqrt{n}))$ mixing of Glauber for periodic/free b.c.

Recall, $\beta_c(\mathbb{Z}^2) = \ln(1 + \sqrt{2})$.

Open: Mixing time for all $+$ boundary for low-temperature region.

Note: FPRAS for Potts $q \geq q_0$ for all β (for periodic boundary)

(Matthew's talk?) [BCHPT '20]

Spatial Mixing

For a box Λ_n and $v \in V$, let $\mathbf{p}(v) = \mathbf{Pr}(v = +)$.

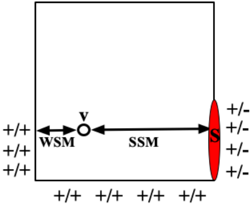
Weak Spatial Mixing (WSM):
$\exists C, \alpha > 0$, all Λ_n, all $v \in V$, all boundaries σ, η on $T \subset \partial\Lambda_n$:

$$|\mathbf{p}^\sigma(v) - \mathbf{p}^\eta(v)| \leq C\exp(-\alpha \mathrm{dist}(v, T)))$$

Strong Spatial Mixing (SSM):
$\exists C, \alpha > 0$, all Λ_n, all $v \in V$, all boundaries σ, τ on $T \subset \partial\Lambda_n$:

$$|\mathbf{p}^\sigma(v) - \mathbf{p}^\tau(v)| \leq C\exp(-\alpha \mathrm{dist}(v, S)),$$

where σ and τ differ on $S \subset T$.



In 2-dimensions, for all $\beta < \beta_c$: SSM holds.

Spatial Mixing

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Strong Spatial Mixing (SSM):
$\exists C, \alpha > 0$, all Λ_n, all $v \in V$, all boundaries σ, τ on $T \subset \partial\Lambda_n$:

$$|\mathbf{p}^{\sigma}(v) - \mathbf{p}^{\tau}(v)| \leq C \exp(-\alpha \mathrm{dist}(v, S)),$$

where σ and τ differ on $S \subset T$.

Pointwise Strong Spatial Mixing: (equivalent to SSM on $\mathbb{Z}^2$)
$\exists C, \alpha > 0$, all Λ_n, all $v \in V$, all $y \in \partial\Lambda_n$, boundaries σ, σ^y:

$$|\mathbf{p}^{\sigma}(v) - \mathbf{p}^{\sigma^y}(v)| \leq C \exp(-\alpha \mathrm{dist}(v, y))$$

In 2-dimensions, for all $\beta < \beta_c$: SSM holds.

Proof idea [MO '94, MOS '94, Cesi '01, DSVW '04]

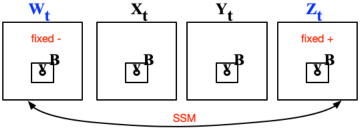
SSM $\Longrightarrow$ $O(n \log n)$ mixing on $L \times L$ box Λ with volume $n = |V|$:

Arbitrary X_0, Y_0. Goal: $\mathbf{Pr}(X_T \neq Y_T) \leq 1/4$ for $T = O(n \log n)$.
Suffices: for all $v \in V$, $\mathbf{Pr}(X_T(v) \neq Y_T(v)) \leq 1/(4n)$.

Boosting argument: Suppose we know $T_{\text{mix}} = n^{100}$.
For $\ell = \log n$, consider small $\ell \times \ell$ box B_v around v.
After $n^{1.02}$ steps on big Λ, $n^{.01}$ updates on small B_v so locally mixed!

Monotonicity: couple so that if $X_t \leq Y_t$ then $X_{t+1} \leq Y_{t+1}$.

Suffices to couple $W_0 =$ all -1 and $Z_0 =$ all $+1$.
Bounding chains: $W_0 = -1, Z_0 = +1$, frozen in $\overline{B}$: $W_t \leq X_t \leq Y_t \leq Z_t$



$\mathbf{Pr}(X_T(v) \neq Y_T(v)) \leq \mathbf{Pr}(W_T(v) \neq Z_T(v))$

$\leq |\mathbf{Pr}(W_T(v) = +) - \mu_B^-(v = +)| + |\mu_B^-(v = +) - \mu_B^+(v = +)| + |\mu_B^+(v = +) - \mathbf{Pr}(Z_T(v) = +)| \leq 1/4n,$

by induction (outer terms) + SSM (inner term).

$T_{\text{mix}}^{\text{new}}(n) = O(n/\log n) \times T_{\text{mix}}^{\text{old}}(C \log^2 n)$:

$$2^n \to n^{O(\log n)} \to O(n^{1+\epsilon}) \to O(n \log^2 n).$$

Ferro Ising on general graphs

What about general graphs? (previous only works for amenable graphs)

For graphs of maximum degree Δ,

computational phase transition at tree critical point.

high temp. $\beta < \beta_c$ $\beta_c(\mathbb{T}_\Delta)$ low temp. $\beta > \beta_c$

$O(n \log n)$ mixing of Glauber on all G of max degree Δ

$\exp(\Omega(n))$ mixing of Glauber on random Δ-regular

Let $\beta_c(\mathbb{T}_d) = \ln\left(\frac{\Delta}{\Delta-2}\right)$ denote the critical point for infinite d-regular tree $\mathbb{T}_d$.

NOW: [Mossel-Sly '13]: All Δ, all $\delta > 0$, $\exists C = C(\Delta, \delta)$, all G max deg Δ, all $\beta < (1-\delta)\beta_c(\mathbb{T}_\Delta)$,

$$T_{\mathrm{mix}} \leq Cn \log n.$$

[Jerrum-Sinclair '93]: $\forall G, \beta$, FPRAS (using high-temp. expansion)
[Guo-Jerrum '17]: $\forall G, \beta$, Swendsen-Wang poly(n) mixing

Spatial Mixing

For $y \in \partial\Lambda_n$ and boundary σ, obtain σ^y by "flipping" spin at y.

Pointwise Strong Spatial Mixing: (equivalent to SSM on $\mathbb{Z}^2$)
Exists $C, \alpha > 0$, all Λ_n, all $v \in V$, all $y \in \partial\Lambda_n$, boundaries σ, σ^y:

$$|\mu_\sigma(v = +) - \mu_{\sigma^y}(v = +)| \leq C \exp(-\alpha \mathrm{dist}(v, y))$$

For $v \in V$ and integer $R \geq 1$, let $B_R(v) = \{w : \mathrm{dist}(v, w) \leq R\}$.

Aggregate Strong Spatial Mixing (ASSM) for graph $G = (V, E)$:
Exists R, all $v \in V$, for $B = B_R(v)$,

$$\text{ASSM holds if } \sum_{y \in \partial B} \max_{\sigma, \sigma^y} |\mu_\sigma(v = +) - \mu_{\sigma^y}(v = +)| \leq \frac{1}{4}.$$

For all G of max degree Δ, all $\beta < \beta_c(\mathbb{T}_\Delta)$, ASSM holds on G.

[Mossel-Sly '09] Proof Approach:

- Previous approach for grid:
  Box/ball B of radius $\Omega(\log n)$ around v.
  But for arbitrary G, $|B|$ can be $|G|$.
  —Use constant R satisfies ASSM.

- Need to do multiple stages, can't couple in one round.
  $\Longrightarrow$ Induction on disagreement probability for a vertex
  Key Lemma: For all $s \geq 0$,

$$\max_v \mathbf{Pr}\left(X_{s+T'}(v) \neq Y_{s+T'}(v)\right) \leq \frac{1}{2} \max_v \mathbf{Pr}\left(X_s(v) \neq Y_s(v)\right),$$

where $T' = C \frac{n}{|B|} T_{\mathrm{mix}}(|B|) = O(n)$. $\Longrightarrow$ Hence, $T_{\mathrm{mix}} = O(n \log(n/\epsilon))$.

- Bounding chains: $W_0 = -1, Z_0 = +1$, but only frozen on $\overline{B}$ for $t > s$.
  $W_s(\overline{B})$ and $Z_s(\overline{B})$ are arbitrary, so no monotonicity.
  [Peres-Winkler '13] Censoring: "Extra moves don't hurt":
  $d_{\mathrm{TV}}(W_t, \mu) \geq d_{\mathrm{TV}}(X_t, \mu)$ and $d_{\mathrm{TV}}(Y_t, \mu) \leq d_{\mathrm{TV}}(Z_t, \mu)$.
  $\Longrightarrow$ suffices to bound $\mathbf{Pr}\left(W_{s+T'}(v) \neq Z_{s+T'}(v)\right)$.

Antiferro Models on General Graphs

What about

Antiferrromagnetic Ising model on graphs of max degree Δ?

- Computational phase transition at tree critical point?
- FPRAS/FPTAS for approximate counting?
- Rapid mixing of Glauber?

Focus on hard-core model

Any 2-spin antiferromagnetic model, e.g. antiferro Ising.

Hard-Core (Gas) Model

For $G = (V, E)$, independent set is $\sigma \subset V$ where:
for all $(y, z) \in E$, $y \notin \sigma$ or $z \notin \sigma$.

Graph $G = (V, E)$, *fugacity* $\lambda > 0$, for each *independent set* σ we have

$$\text{Gibbs distribution:} \qquad \mu(\sigma) = \frac{\lambda^{|\sigma|}}{Z}$$

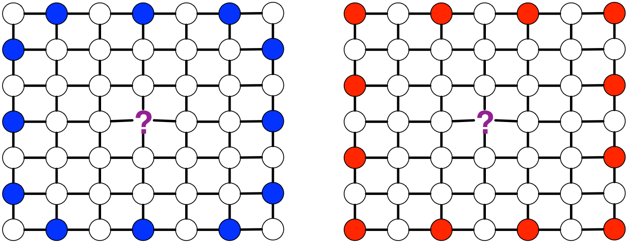
where

$$\text{Partition function:} \qquad Z = \sum_{\sigma} \lambda^{|\sigma|}$$

$\lambda = 1$, $Z = |\Omega| = \#$ of independent sets.

Inuition: Small λ easier: for $\lambda < 1$ prefer empty set/smaller sets.
Large λ harder: for $\lambda > 1$ prefer max IS's/larger sets.

Influence of boundary:

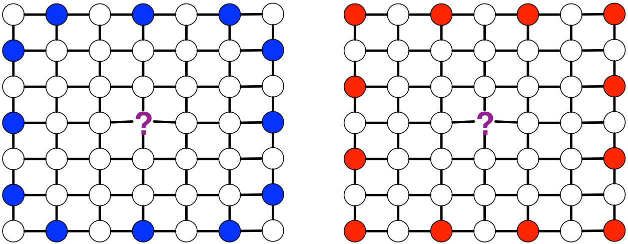


Let $\mathrm{p}_L^{\mathrm{even}} = \mathbf{Pr}(\text{Origin occupied } \mid \text{even boundary for } L \times L \text{ box})$.

Let $\mathrm{p}_L^{\mathrm{odd}} = \mathbf{Pr}(\text{ Origin occupied } \mid \text{odd boundary for } L \times L \text{ box})$.

Hard-core Phase Transition

Influence of boundary:



Conjecture: There exists critical point $\lambda_c(\mathbb{Z}^2)$ where:

For all $\lambda < \lambda_c(\mathbb{Z}^2)$, $\lim_{L\to\infty} \mathrm{p}_L^{\mathrm{even}} - \mathrm{p}_L^{\mathrm{odd}} = 0$ uniqueness

For all $\lambda > \lambda_c(\mathbb{Z}^2)$, $\lim_{L\to\infty} \mathrm{p}_L^{\mathrm{even}} - \mathrm{p}_L^{\mathrm{odd}} > 0$ non-uniqueness

For 2-dimensional integer lattice $\mathbb{Z}^2$:

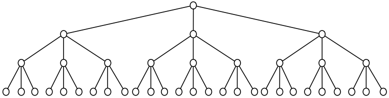
Conjecture: $\lambda_c(\mathbb{Z}^2) \approx 3.79$

Best bounds: $2.53 < \lambda_c(\mathbb{Z}^2) < 5.36$ [SSSY '15, BGRT '13]

Phase Transition on Trees

For Δ-regular tree of height ℓ:

$$\text{Let } p_\ell := \mathbf{Pr}\,(\text{root is occupied})$$



Extremal cases: even versus odd height.

$$\text{Does } \lim_{\ell\to\infty} p_{2\ell} = \lim_{\ell\to\infty} p_{2\ell+1} \; ?$$

$\lambda_c(\mathbb{T}_\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^\Delta} \approx \frac{e}{\Delta-2}$. [Kelly '86]

$\lambda \le \lambda_c(\mathbb{T}_\Delta)$: No boundary effects root. uniqueness

$\lambda > \lambda_c(\mathbb{T}_\Delta)$: Exist boundaries effect root. non-uniqueness

Tree/BP recursions: $p_{\ell+1} = \frac{\lambda(1-p_\ell)^{\Delta-1}}{1+\lambda(1-p_\ell)^{\Delta-1}}$

Key: Unique vs. Multiple fixed points of 2-level recursions.

Antiferromagnetic on general graphs

$$\lambda_c(\mathbb{T}_\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}} \approx \frac{e}{\Delta-2}.$$

high temp. $\lambda < \lambda_c$ — $\lambda_c(\mathbb{T}_\Delta)$ — low temp. $\lambda > \lambda_c$

$O(n\log n)$ mixing of Glauber on all G of max degree Δ — No FPRAS

NOW: **1.** *Theorem [Weitz '06]:* For all Δ, all $\delta > 0$, exists $C = C(\Delta, \delta)$, for $\lambda < (1-\delta)\lambda_c(\mathbb{T}_\Delta)$ and G of max degree Δ, can approximate Z_G within $(1 \pm \epsilon)$ in time $(n/\epsilon^2)^C$.

[Barvinok '16,Patel-Regts '17,Peters-Regts '19]: Alternative FPTAS via Barvinok's polynomial interpolation method. (Next talk?)

2. Theorem [Chen-Liu-V '20]: For all Δ, all $\delta > 0$, exists $C = C(\Delta, \delta)$, for $\lambda < (1-\delta)\lambda_c(\mathbb{T}_\Delta)$ and G max degree Δ, Glauber mixes in time $\leq Cn\log n$.
(Uses [Anari-Liu-Oveis Gharan '20] Spectral Independence approach.)

Theorem [Sly '09, Sly-Sun '14, Galanis-Stefankovic-V '14]: For all $\lambda > \lambda_c(\delta)$, unless $NP = RP$, no FPRAS for all graphs of maximum degree Δ.
(Hard to approximate within C^n for $C = C(\Delta)$).

[Weitz '06]'s algorithmic approach

Idea for FPTAS for $Z_G(\lambda)$.

Input: $G = (V, E)$ of max degree Δ and $\lambda < \lambda_c(\mathbb{T}_\Delta)$.

Fix (arbitrarily) a vertex v, then:

1. Compute marginal prob. v is unoccupied/occupied;
2. Recurse on $G \setminus v$ or $G \setminus (v \cup N(v))$ with probabilities from step 1.

[Weitz '06]'s algorithmic approach

Idea for FPTAS for $Z_G(\lambda)$.

Input: $G = (V, E)$ of max degree Δ and $\lambda < \lambda_c(\mathbb{T}_\Delta)$.

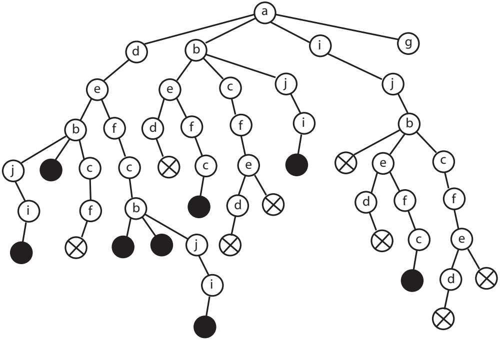
Fix (arbitrarily) a vertex v, then:

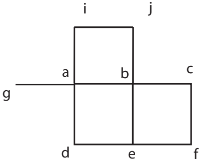
1. Compute marginal prob. v is unoccupied/occupied; HOW?
2. Recurse on $G \setminus v$ or $G \setminus (v \cup N(v))$ with probabilities from step 1.

WEITZ'S SAW TREE

Fix $G = (V, E)$ and $a \in V$.
Let $T = T_{\text{saw}}(G, a)$ be the *self-avoiding walks* in G starting at a,
with a *particular fixed assignment to the leaves* of T.

Theorem [Weitz '06]: $\mathbf{Pr}_{\sigma \sim \mu_G}(a \notin \sigma) = \mathbf{Pr}_{\sigma \sim \mu_T}(a \notin \sigma)$





Boundary: for each vertex order neighbors.
Root-leaf path ends with a cycle, e.g., $b - c - f - e - b$.
Then fix the leaf to unoccupied if $c > e$ and occupied if $c < e$.

High-level idea of [Weitz '06]'s approach

Let $T = T_{\mathrm{saw}}(G, v)$ be the *self-avoiding walks* in G starting at v, with a *particular fixed assignment to the leaves* of T.

Theorem [Weitz '06]: $\mathbf{Pr}_{\sigma \sim \mu_G}(v \notin \sigma) = \mathbf{Pr}_{\sigma \sim \mu_T}(v \notin \sigma)$
(Only holds for 2-spin systems.)

In tree of size N, compute marginal of root (tree recursions) in poly(N) time but SAW tree $N = \Delta^{O(n)}$.

Second ingredient:
For every tree T of max deg Δ, SSM holds when $\lambda < \lambda_c(\mathbb{T}_\Delta)$.
$\Longrightarrow$ truncate tree at depth $O(\log n)$
Running time: $\Delta^{O(\log n)} = n^{O(\log \Delta)}$.

How to prove SSM?
Show contraction of a potential function for the Jacobian of the log ratio of marginals (tree recursions).

[Li-Lu-Yin '13] 2-spin antiferro. spin system in tree uniqueness region.

ANTIFERROMAGNETIC ON GENERAL GRAPHS

$$\lambda_c(\mathbb{T}_\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}} \approx \frac{e}{\Delta-2}.$$

high temp. $\lambda < \lambda_c$ — $\lambda_c(\mathbb{T}_\Delta)$ — low temp. $\lambda > \lambda_c$

$O(n \log n)$ mixing of Glauber on all G of max degree Δ — No FPRAS

1. *Theorem [Weitz '06]:* For all Δ, all $\delta > 0$, exists $C = C(\Delta, \delta)$, for $\lambda < (1-\delta)\lambda_c(\mathbb{T}_\Delta)$ and G of max degree Δ, can approximate Z_G within $(1 \pm \epsilon)$ in time $(n/\epsilon^2)^C$.

NOW: **2.** *Theorem [Chen-Liu-V '20]:* For all Δ, all $\delta > 0$, exists $C = C(\Delta, \delta)$, for $\lambda < (1-\delta)\lambda_c(\mathbb{T}_\Delta)$ and G max degree Δ, Glauber mixes in time $\leq Cnlogn$.
(Uses [Anari-Liu-Oveis Gharan '20] Spectral Independence approach.)
(see Zongchen's talk for more details)

ALO's Spectral Independence

[Anari-Liu-Oveis Gharan '20] Spectral Independence approach:

For a pair of vertices $u, w \in V$, the influence of u on w:

$$\mathcal{I}_\mu(u \to w) = \mu(\sigma(w) = 1 | \sigma(u) = 1) - \mu(\sigma(w) = 1 | \sigma(u) = 0)$$

We need to consider the influence for any boundary or pining:
For $\Lambda \subset V$ and $\tau \in \Omega_\Lambda$, let

$$\mathcal{I}_\mu^\tau(u \to w) = \mu(\sigma(w) = 1 | \sigma(u) = 1, \tau) - \mu(\sigma(w) = 1 | \sigma(u) = 0, \tau)$$

Let Ψ_μ^τ denote the $n \times n$ *Influence matrix* of pairwise influences.

Definition: η-spectrally independent if for all $\Lambda \subset V$, all $\tau \in \Omega_\Lambda$, $\lambda_1(\Psi_\mu^\tau) \leq \eta$.
Definition: μ is *b-marginally bounded* if for all $\Lambda \subset V$, all $\tau \in \Omega_\Lambda$, all $u \in V \setminus \Lambda$, all $i \in \Omega_u^\tau$, $\mu(\sigma_u = i | \tau) \geq b$.

Theorem [Chen-Liu-V '20]: If b-marginally bounded and η-spectrally independent then $\exists C = C(\Delta, \eta, b)$, the mixing time of Glauber is $\leq Cn \log n$.

Note, $C = \left(\frac{\Delta}{b}\right)^{O(\eta/b^2)}$.

For hard-core: show $\eta = O(1/\delta)$ and $b = \Omega(\lambda/(1+\lambda)^\Delta))$.

Bounding Influence Matrix

Pairwise influence:

$$\mathcal{I}_\mu(u \to w) = \mu(\sigma(w) = 1 | \sigma(u) = 1) - \mu(\sigma(w) = 1 | \sigma(u) = 0)$$

Let Ψ_μ^τ denote the $n \times n$ *Influence matrix* of pairwise influences.

How to bound $\lambda_1(\Psi) \leq \eta$?

Note: $\lambda_1(\Psi) \leq \max_{r \in V} \sum_{v \in V} |\mathcal{I}(v \to r)|$ and $\lambda_1(\Psi) \leq \max_{r \in V} \sum_v |\mathcal{I}(r \to v)|$.

Key Lemma: Fix $\Lambda \subset V, \tau \in \Omega_\Lambda$. Fix $r \in V$, let $T = T_{\text{saw}}(G, r, \tau)$.

$$\mathcal{I}_G^\tau(r \to w) = \sum_{\widehat{w} \in S_w} \mathcal{I}_T^\tau(r \to \hat{w}),$$

where S_w is the set of all copies of w in T.
Then, $\sum_{w \in V} |\mathcal{I}_G^\tau(r \to w)| = \sum_\ell \sum_{z \in L_\ell} |\mathcal{I}_T(r \to z)|$,
where L_ℓ are the vertices at distance ℓ from the root r in T.

Finally, for ferro Ising can bound using ASSM (spatial mixing in [Mossel-Sly]), and for other 2-spin systems using potential functions as for proofs of SSM.

Conclusions

Open Problems:

- Mixing time is $\leq Cn\log n$ where $C = C(\Delta, \delta)$.
  For Ising obtain $C = \text{poly}(\Delta)$.
  Can we establish it for hard-core model?
- SSM $\overset{?}{\Longleftrightarrow}$ Spectral Independence
- Absence of complex zeros $\overset{?}{\Longleftrightarrow}$ Spectral Independence
  (Anari's talk?)
- Ferro Potts:
  Open: FPRAS for $\beta < \beta_u(\mathbb{T}_\Delta)$?
  Known: #BIS-hard for $\beta > \beta_c(\mathbb{T}_\Delta)$ [GSVY '14]
- k-colorings:
  Known: $O(n\log n)$ mixing when $k > 2\Delta$ [Jerrum '95]
  $O(n^2)$ mixing when $k > (11/6 - \epsilon)\Delta$ [CDMPP'19]
  $O(n\log n)$ for triangle-free graphs $k > 1.764\Delta$ [FGYZ'20,CGSV'20,CLY'20]
  For even $k < \Delta$, no FPRAS (unless $NP = RP$) [GSV'14]
  Open: FPRAS for $k > \Delta + 1$?